Compensator Design to Improve Steady-State Error Using Root Locus

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These notes are lecture notes prepared by Prof. Guy Beale for presentation in ECE 421, *Classical Systems and Control Theory*, in the Electrical and Computer Engineering Department, George Mason University, Fairfax, VA. Additional notes can be found at: http://teal.gmu.edu/~gbeale/examples.html.

I. INTRODUCTION

The purpose of compensator design generally is to satisfy both transient and steady-state specifications. In the root locus design approach presented here, these two tasks are approached separately. First, the transient performance specifications are satisfied, using one or more stages of lead (usually) or lag compensation. Once that is accomplished, the steady-state error can be dealt with if necessary.

The terminology that I use for the compensator designed using root locus methods to satisfy the steady-state error specification is "special" lag compensator. It will always be a lag compensator, and it is "special" in the sense that it has the mission of reducing steady-state error without having any effect on the transient performance compensation that has already been done. Therefore, this special lag compensator is not supposed to reshape the root locus as would be done to satisfy transient performance specifications.

Conceptually, the design procedure presented here is graphical in nature. Only very simple calculations are needed to design the special lag compensator, and the procedure can be easily automated in MATLAB along with other compensator design functions.

The primary references for the procedures described in these notes are [1]–[3]. Other references that contain similar material are [4]–[11].

II. DESIGN PROCEDURE

A. Compensator Structure

The basic special lag compensator consists of one pole and one zero. Multiple stages may be used if the required amount of reduction in steady-state error is too large to be accomplished by a single stage. The compensator structure is described in more detail in the notes[1] *Compensator Design to Improve Transient Performance Using Root Locus*. Assuming the circuit implementation shown in Ogata [3], the transfer function for the special lag compensator is

$$G_{c\_spec\_lag}(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{(s - z_c)}{(s - p_c)} = \frac{\alpha\left(\dfrac{s}{-z_c} + 1\right)}{\left(\dfrac{s}{-p_c} + 1\right)} = \frac{\alpha\left(\dfrac{s}{|z_c|} + 1\right)}{\left(\dfrac{s}{|p_c|} + 1\right)} \tag{1}$$

where the last form of the compensator comes from realizing that the pole and zero of the special lag compensator will be in the left-half of the s-plane.

Two points can be noted about the transfer function in (1). The first is that the gain $K_c = 1$. This reduces the number of parameters that must be computed from three to two, and it means that the absolute value of the gain in the second operational amplifier in the circuit implementation of the compensator will be $R_4/R_3 = C_2/C_1$.

The second, and more important, point to note is that the steady-state error of the given system (plant and transient-response compensator) will be reduced by a factor of α by the special lag compensator. This can be seen by taking the limit of $G_{c\_spec\_lag}(s)$ as $s \to 0$. This means that once we know how much the steady-state error must be reduced, we also know the ratio of compensator zero to pole.

B. Outline of the Procedure

The following steps outline the procedure that will be used to design the special lag compensator using root locus methods in order to satisfy steady-state specifications. Prior to carrying out these steps, it is assumed that all transient response specifications have been satisfied. Compensator design to satisfy those requirements is discussed in separate notes, *Compensator Design to Improve Transient Performance Using Root Locus*. It is also assumed that at this point in the design the compensated system has the correct System Type. Therefore, the only task that remains is to numerically satisfy a steady-state error specification for the particular type of reference input signal that corresponds to the System Type.

1) Calculate the value of the steady-state error for the system $G_{c\_trans}(s)G_p(s)$, where the transfer function $G_{c\_trans}(s)$ is the compensator designed to satisfy transient performance specifications and adjust the System Type if necessary.
2) Calculate the ratio of the actual steady-state error to the desired value for the error. This ratio also becomes the ratio $\alpha = z_c/p_c$.
3) Design the compensator:
    a) Place the zero of the special lag compensator to the right of the real-axis projection of the dominant closed-loop pole by a factor of 50–100.
    b) Place the compensator pole to the right of the zero by a factor of α.
4) If necessary, choose appropriate resistor and capacitor values to implement the compensator design.

[1] See http://teal.gmu.edu/~gbeale/ece_421/examples_421.html

To illustrate the design procedure, the following system model and specifications will be used:

$$G_p(s) = \frac{0.375(s+0.8)}{s(s+0.2)(s+1)(s+1.5)} \tag{2}$$

- steady-state error specification for a unit ramp input is $e_{ss\_specified} = 0.2$;
- step response settling time specification is $T_{s-specified} \approx 16$ seconds;
- step response overshoot specification $PO_{specified} \approx 20\%$.

The two transient response specifications can be satisfied by the phase lead compensator

$$G_{c\_lead}(s) = \frac{1.5192\,(s+0.2)}{(s+0.6583)} \tag{3}$$

Thus, the open-loop forward path transfer function that provides the starting point for the design of the special lag compensator is

$$\begin{aligned} G(s) &= G_{c\_lead}(s)G_p(s) = \frac{1.5192\,(s+0.2)}{(s+0.6583)} \cdot \frac{0.375(s+0.8)}{s(s+0.2)(s+1)(s+1.5)} \\ &= \frac{0.5697(s+0.8)}{s(s+0.6583)(s+1)(s+1.5)} \end{aligned} \tag{4}$$

Figure 1 shows the uncompensated and lead-compensated root locus plots and closed-loop step responses. The point $s = s_1$ shown in the root locus plots is the desired closed-loop pole location $s_1 = -0.25 + j0.488$ that was used to design the phase lead compensator. With the lead-compensated transfer function in (4), the step response has an overshoot of 19.4% and a settling time of 16.3 seconds, so the transient specifications are satisfied. The closed-loop poles are located at $s = -0.25 \pm j0.488$, -0.2, -0.8285, -1.8298.

C. Determining the System's Steady-State Error

The first thing that must be done after the transient response specifications have been satisfied is to determine the steady-state error of the compensated system. Defining the number of open-loop poles of a system that are located at $s = 0$ to be the System Type N, and restricting the reference input signal to having Laplace transforms of the form $R(s) = A/s^q$, the steady-state error and error constant are (assuming that the closed-loop system is bounded-input, bounded-output stable)

$$e_{ss} = \lim_{s \to 0} \left[\frac{As^{N+1-q}}{s^N + K_x} \right] \tag{5}$$

where

$$K_x = \lim_{s \to 0} \left[s^N G(s) \right] \tag{6}$$

The transfer function $G(s)$ appearing in (6) in our example is the transfer function $G(s)$ in (4), so the system is Type 1, and we know that the closed-loop system is stable. Therefore, (5) and (6) can be used to determine the steady-state error of the lead-compensated system. From (6) we have

$$\begin{aligned} K_x &= \lim_{s \to 0} \left[s \cdot \frac{0.5697(s+0.8)}{s(s+0.6583)(s+1)(s+1.5)} \right] \\ &= 0.4615 \end{aligned} \tag{7}$$

The steady-state error of the lead-compensated system for a ramp input $(q = 2)$ is computed from (5) to be

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} \left[\frac{s^0}{s + 0.4615} \right] = \frac{1}{0.4615} \\ &= 2.1666 \end{aligned} \tag{8}$$

This value is compared with the specified value to determine whether or not the special lag compensator needs to be designed. If the value of steady-state error computed from (5) is larger than the specified value, as it is in this example, then the special lag compensator must be designed. If the computed value is less than or equal to the specified value, no additional compensation is needed. It will always be assumed that if transient response specifications are satisfied, then any value for steady-state error less than or equal to the maximum allowed (specified) value is acceptable. If there is no specification on steady-state error, then this procedure may be omitted completely.

For this example, we have $e_{ss} = 2.1666$ and $e_{ss\_specified} = 0.2$. Therefore, the special lag compensator is needed in order to satisfy the specification.

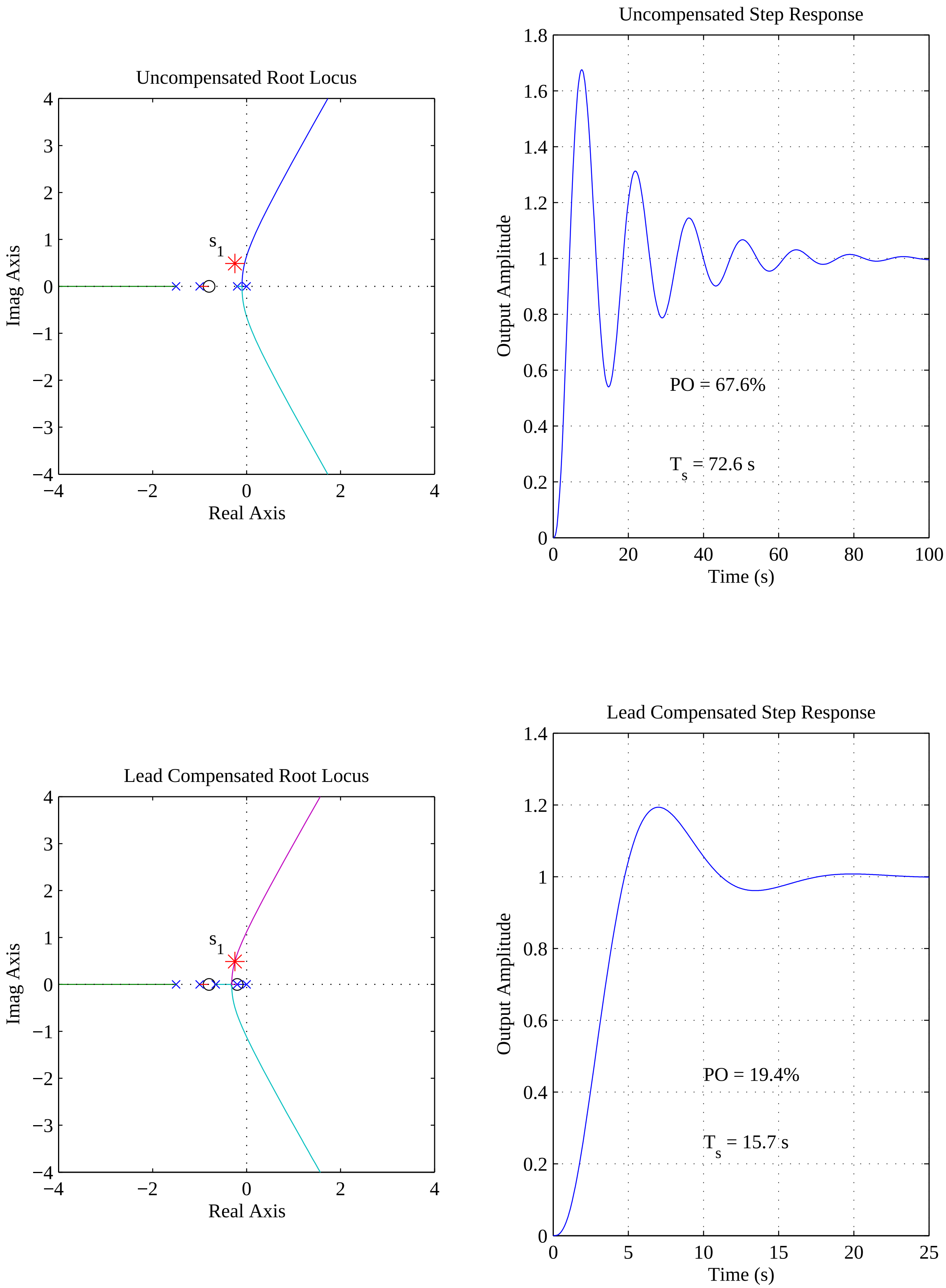

Fig. 1. Root locus and step responses for the uncompensated and lead-compensated systems.

D. Determining the Value of α

Once it has been determined that the special lag compensator is needed, a simple calculation provides the value for $\alpha = z_c/p_c$. Using the transfer function in (1) for the compensator, the steady-state error for the system will be reduced by the special lag compensator by an amount

$$\begin{aligned} K_{x\_spec\_lag} &= \lim_{s\to 0} \left[\frac{\alpha\left(\dfrac{s}{-z_c}+1\right)}{\left(\dfrac{s}{-p_c}+1\right)} \right] \\ &= \alpha = \frac{z_c}{p_c} > 1 \end{aligned} \tag{9}$$

Thus, the special lag compensator will reduce the steady-state error present in $G(s)$ by the ratio of the compensator zero to pole. This gives the following expression for α.

$$\alpha = \frac{e_{ss\_actual}}{e_{ss\_specified}} = \frac{K_{x\_required}}{K_{x\_actual}} \tag{10}$$

where K_{x_actual} is the value of K_x for $G_{c\_trans}(s)G_p(s)$, and the last form of the expression assumes that $G(s)$ is Type 1 or higher so that $e_{ss} = 1/K_x$. If $G(s)$ is Type 0, then the last form in (10) would be changed to $(1 + K_{x\_required}) / (1 + K_{x\_actual})$.

For the system $G(s)$ defined in (4), the steady-state error for a ramp input was computed to be $e_{ss} = 2.1666$, so with the specified value $e_{ss\_specified} = 0.2$, the amount of error reduction and the ratio of compensator zero to pole are both given by

$$\alpha = \frac{z_c}{p_c} = \frac{2.1666}{0.2} = 10.833 \tag{11}$$

In terms of only satisfying the steady-state error specification, the pole and zero of the lag compensator may be placed anywhere on the real axis as long as two requirements are met. The first is the ratio of compensator zero to pole is given by α. The second requirement is that closed-loop stability is maintained; otherwise there is no steady-state response of the system.

E. Placing the Compensator Zero and Pole

Although the steady-state error specification can be satisfied as long as the two requirements in the previous paragraph are met, we also have the goal of maintaining the desired transient behavior. Some choices for z_c and p_c in the special lag compensator would significantly change the transient performance even though closed-loop stability and steady-state error were not effected.

Assume that a point $s = s_1$ was chosen to be a desired dominant closed-loop pole and that a compensator has been designed so that point actually is a closed-loop pole and transient performance specifications are satisfied. As previously mentioned, this is the starting point for the design of the special lag compensator. With this being the case, we have the conditions

$$\begin{aligned} G(s) &= G_{c\_trans}(s)G_p(s) \\ |G(s_1)| &= 1, \qquad \angle G(s_1) = 180^\circ\,(2l+1) \end{aligned} \tag{12}$$

The final forward path transfer function will be the series combination of the special lag compensator and the transfer function in (12). For the point s_1 to remain a closed-loop pole, we need the conditions

$$\begin{aligned} G_{final}(s) &= G_{c\_spec\_lag}(s)G(s) = G_{c\_spec\_lag}(s)G_{c\_trans}(s)G_p(s) \\ |G_{final}(s_1)| &= 1, \qquad \angle G_{final}(s_1) = 180^\circ\,(2l+1) \end{aligned} \tag{13}$$

By comparing (12) and (13), we can see that the magnitude and phase requirements of the special lag compensator at the point s_1 are

$$\begin{aligned} |G_{c\_spec\_lag}(s)| &= 1 = \frac{|s_1 - z_c|}{|s_1 - p_c|} \\ \angle G_{c\_spec\_lag}(s) &= 0^\circ = \angle(s_1 - z_c) - \angle(s_1 - p_c) \end{aligned} \tag{14}$$

The two expressions in (14) indicate that vectors drawn from the pole and zero to the point s_1 must have the same length and that they must make the same angles with respect to the positive real axis. Trying to satisfy the conditions in (14) exactly would result in z_c and p_c being placed at the same point which would violate the requirement that they be separated by a

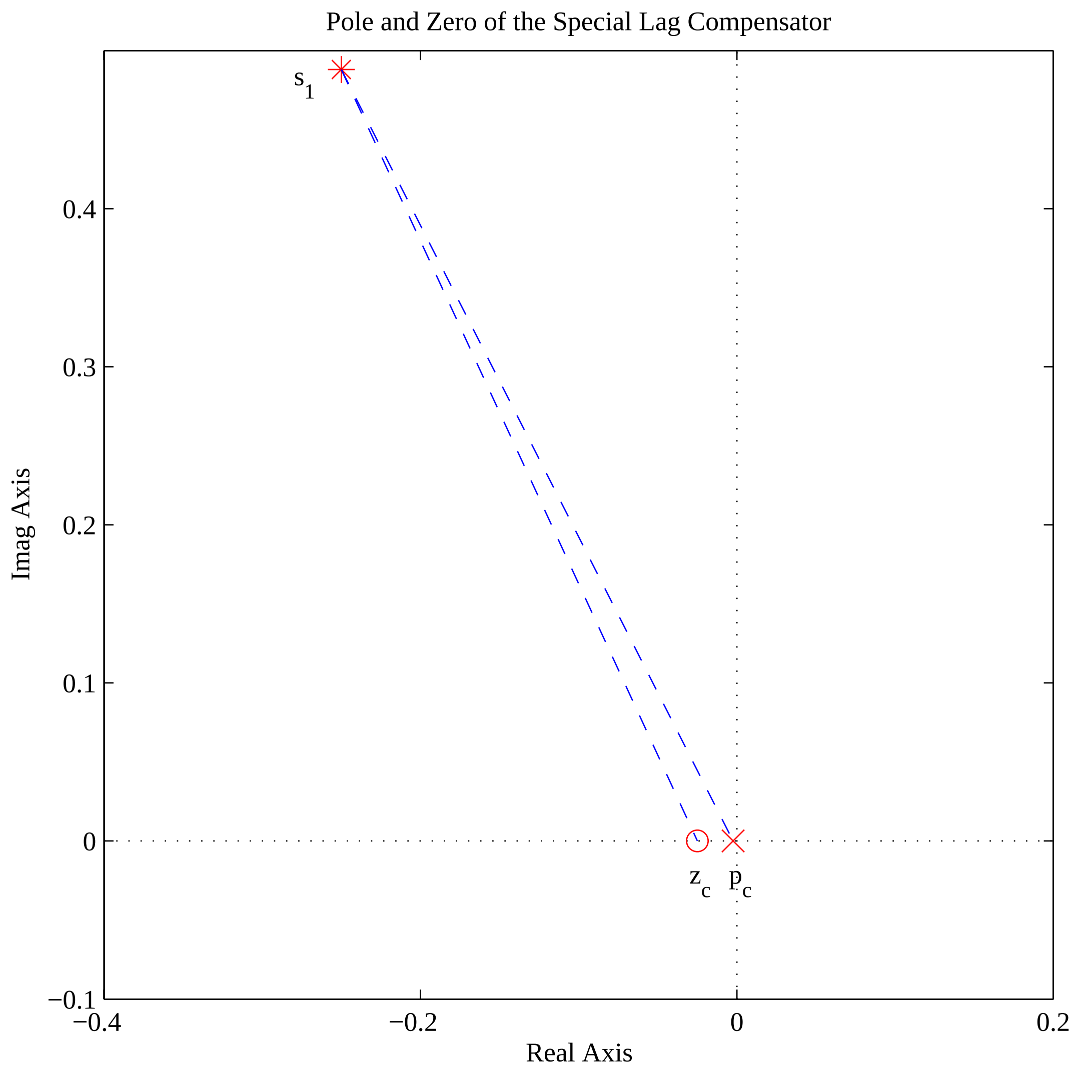

Fig. 2. Illustration of placing the zero and pole of the special lag compensator to maintain s_1 as a closed-loop pole.

factor of $\alpha > 1$. The only region in the s-plane that will provide a good approximation to the conditions in (14) while still allowing that $z_c/p_c = \alpha$ is near the origin (at least relative to s_1). Approximating the conditions of (14) means that s_1 will no longer be exactly on the root locus, but a closed-loop pole in the final system should be very near s_1. Figure 2 shows this situation for our example, with $s_1 = -0.25 + j0.488$ and $\alpha = 10.833$. The compensator zero has been placed to the right of the real-axis projection of s_1 by a factor of 10 in this figure. It can be seen that the vectors from z_c and p_c to s_1 have approximately the same lengths and make approximately the same angles with the respect to the axis.

Therefore in order to obtain a good approximation to the conditions of (14), the zero and pole of the special lag compensator will be placed at

$$z_c = \frac{\mathrm{Re}\,[s_1]}{M}, \qquad p_c = \frac{z_c}{\alpha} \tag{15}$$

where M is a suitably large number, generally in the range 50–100.

The larger the value of M, the more closely the conditions in (14) will be met. However, making M extremely large will have negative effects of the implementation of the compensator, as illustrated in the next section. The following table shows the locations of the zero and pole for various values of M for our example system. Also shown are the magnitudes and phases of the special lag compensator at s_1 and the final location of the closed-loop pole closest to s_1.

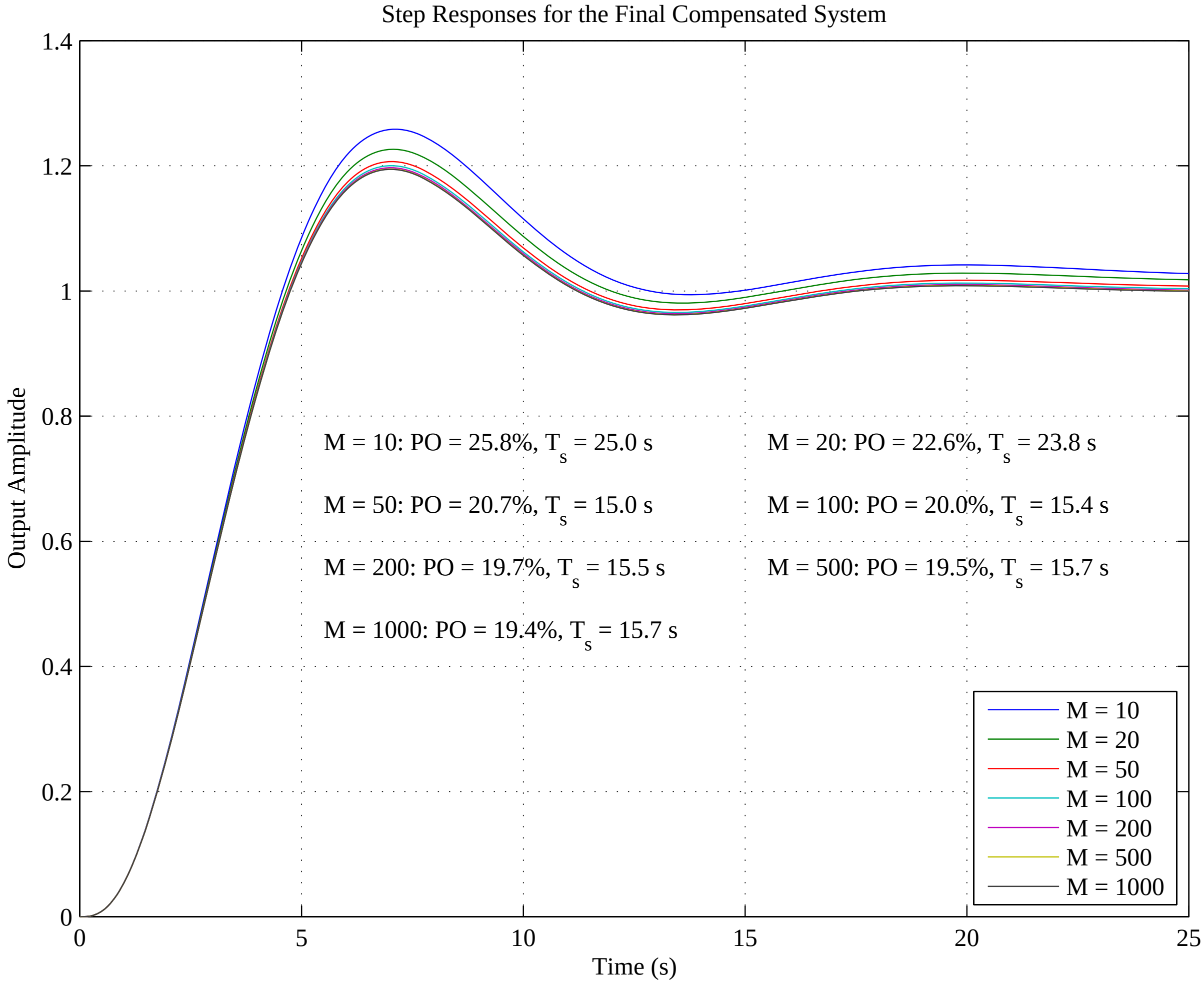

Fig. 3. Closed-loop step responses for various designs of the special lag compensator.

M	z_c	p_c	$\lvert G_{c\_spec\_lag}(s_1)\rvert$	$\angle G_{c\_spec\_lag}(s_1)$	pole
10	$-2.5 \cdot 10^{-2}$	$-2.31 \cdot 10^{-3}$	0.9819	$-2.158°$	$-0.2389 + j0.4784$
20	$-1.25 \cdot 10^{-2}$	$-1.15 \cdot 10^{-3}$	0.9908	$-1.067°$	$-0.2446 + j0.4832$
50	$-5 \cdot 10^{-3}$	$-4.62 \cdot 10^{-4}$	0.9963	$-0.424°$	$-0.2479 + j0.4861$
100	$-2.5 \cdot 10^{-3}$	$-2.31 \cdot 10^{-4}$	0.9981	$-0.212°$	$-0.2490 + j0.4870$
200	$-1.25 \cdot 10^{-3}$	$-1.15 \cdot 10^{-4}$	0.9991	$-0.106°$	$-0.2495 + j0.4875$
500	$-5 \cdot 10^{-4}$	$-4.62 \cdot 10^{-5}$	0.9996	$-0.042°$	$-0.2498 + j0.4878$
1000	$-2.5 \cdot 10^{-4}$	$-2.31 \cdot 10^{-5}$	0.9998	$-0.021°$	$-0.2499 + j0.4879$

It is clear from the table that the closer the zero and pole are to the origin, the more closely the compensator approximates the conditions of (14). Realistically, $M \geq 50$ should be sufficient to have the transient performance of the complete system be close enough to the performance obtained before the design of the special lag compensator. Figure 3 shows the final closed-loop step responses for each of the pole/zero combinations in the table above. The settling times for $M = 10$ and $M = 20$ are larger that the specification allows; the others are all between 15.0 and 15.7 seconds, so that specification is still satisfied for $M \geq 50$. Moreover, the overshoot for $M = 10$ is 25.8%, and for $M = 20$ the overshoot is 22.6%. For $M \geq 50$, the overshoot ranges from 20.7% to 19.4%, so those values of M allow the overshoot specification to be satisfied. The value that I generally use is $M = 100$ since that provides good performance, and it is easy to compute the value of the compensator zero.

Including the special lag compensator in the system introduces an additional closed-loop pole. This pole will be much farther to the right than s_1. However, that pole will not dominate the step response because it is very close to a closed-loop zero (the special lag compensator's open-loop zero). The additional pole will increase the time it takes for the output to look like it actually equals the final value, but the $\pm 2\%$ settling time will generally not be effected. Figure 4 shows the root locus plot for the final system with $M = 100$. The plot is essentially identical to the one for the lead-compensated system in Figure 1, except near the origin. The point s_1 is seen to lie virtually on the root locus. The zoomed view of the root locus shows the

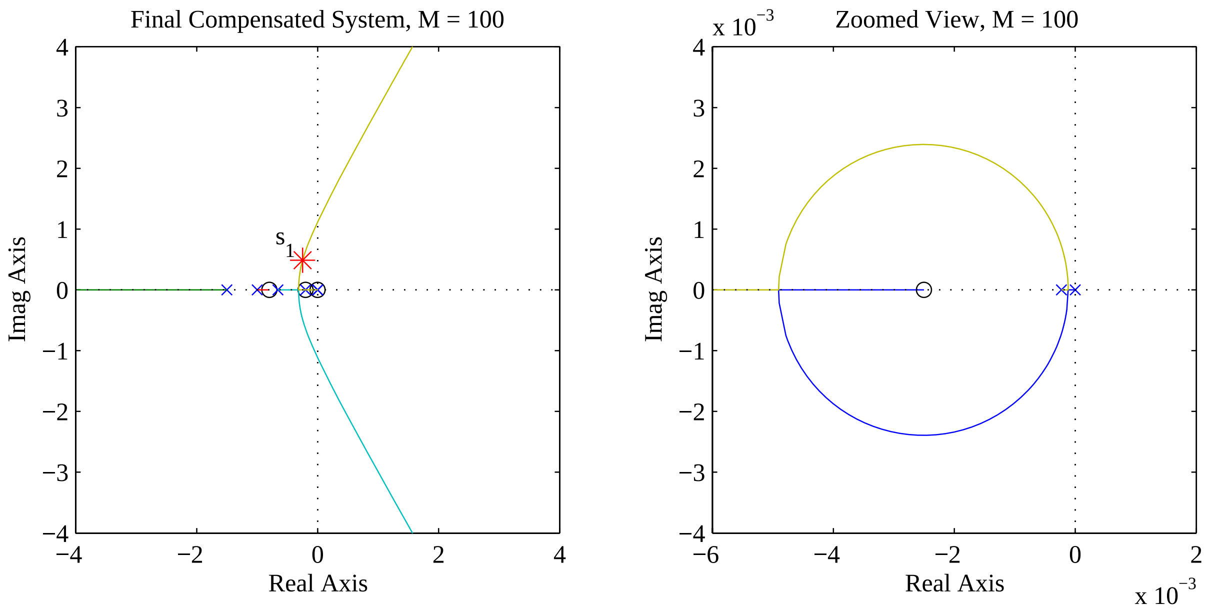

Fig. 4. Root locus for the final compensated system with $M = 100$.

zero and pole of the special lag compensator, as well as the plant pole at $s = 0$. The complex part of the root locus in that region will be a circle centered at the compensator zero (for all practical purposes). From the break-in point to the left of the zero, one closed-loop pole moves to the right toward the zero. This is the additional closed-loop pole mentioned above. The other pole moves from the break-in point to the left toward the lead compensator's pole at $s = -0.6583$.

The special lag compensator designed using (1), (10), and (15) will satisfy the steady-state error specification without disturbing the transient behavior too much, particularly for $M \geq 50$. Remember that making sure that $G(s)$ has the correct System Type should be done at the beginning of the design of the transient performance compensator. In this way, the special lag compensator can be easily designed to satisfy the numerical specification.

If α computed from (10) is large, for example $\alpha > 20$, it may be desirable to implement the compensator in multiple stages in order to restrict the range of component values. The simplest approach is to plan on identical stages, with repeated poles and zeros for the compensator. In this case, each stage of the compensator would have

$$\alpha_{stage} = \frac{z_c}{p_c} = \sqrt[N_{stage}]{\alpha} \tag{16}$$

the zero and pole would be located at

$$z_c = \frac{\mathrm{Re}\,[s_1]}{M}, \qquad p_c = \frac{z_c}{\alpha_{stage}} \tag{17}$$

and the special lag compensator would be

$$G_{c\_spec\_lag} = \frac{(s - z_c)^{N_{stage}}}{(s - p_c)^{N_{stage}}} \tag{18}$$

For our example, using $M = 100$ and $N_{stages} = 2$, we would have

$$\alpha_{stage} = 3.2914, \qquad z_c = -2.5 \cdot 10^{-3}, \qquad p_c = -7.60 \cdot 10^{-4} \tag{19}$$

F. Resistor and Capacitor Values

Based on the usual electronic implementation of the compensator in [3], the circuit for the special lag compensator is the series combination of two inverting operational amplifiers. The first amplifier has an input impedance that is the parallel combination of resistor R_1 and capacitor C_1 and a feedback impedance that is the parallel combination of resistor R_2 and capacitor C_2. The second amplifier has input and feedback resistors R_3 and R_4, respectively. As mentioned in Section II-A, the fact that $K_c = 1$ in the special lag compensator imposes the constraint that $R_4/R_3 = C_2/C_1$.

The impedances in the input and feedback paths of the first op amp depend on the values of the compensator's pole and zero. Specifically,

$$\tau_1 = R_1C_1 = -1/z_c, \qquad \tau_2 = R_2C_2 = -1/p_c, \qquad \alpha = \frac{R_2C_2}{R_1C_1} = \frac{\tau_2}{\tau_1} \tag{20}$$

Therefore, a large value for α results in a wide range of component values. For the example system considered in these notes, with $\alpha = 10.833$, assume that $C_1 = C_2 = 100\ \mu\text{F} = 10^{-4}$ F. Then the following values for the resistors are needed.

M	τ_1 (sec)	τ_2 (sec)	R_1 (Ω)	R_2 (Ω)
10	40	$4.33 \cdot 10^2$	$4 \cdot 10^5$	$4.33 \cdot 10^6$
20	80	$8.67 \cdot 10^2$	$8 \cdot 10^5$	$8.67 \cdot 10^6$
50	200	$2.17 \cdot 10^3$	$2 \cdot 10^6$	$2.17 \cdot 10^7$
100	400	$4.33 \cdot 10^3$	$4 \cdot 10^6$	$4.33 \cdot 10^7$
200	800	$8.67 \cdot 10^3$	$8 \cdot 10^6$	$8.67 \cdot 10^7$
500	2000	$2.17 \cdot 10^4$	$2 \cdot 10^7$	$2.17 \cdot 10^8$
1000	4000	$4.33 \cdot 10^4$	$4 \cdot 10^7$	$4.33 \cdot 10^8$

The table clearly shows that large component values are needed for implementing lag compensators. The closer the zero and pole are to the origin, the larger those values would be. For this reason, M should not be larger than necessary in (15) and multiple stages of compensation should be considered.

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