Digital Control Systems
State Feedback Control

Illustrating the Effects of Closed-Loop Eigenvalue Location and Control Saturation for a Stable Open-Loop System

GEORGE MASON UNIVERSITY

Continuous-Time System

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 0.6s + 1}, \qquad \zeta = 0.3, \omega_n = 1\, r/s$$

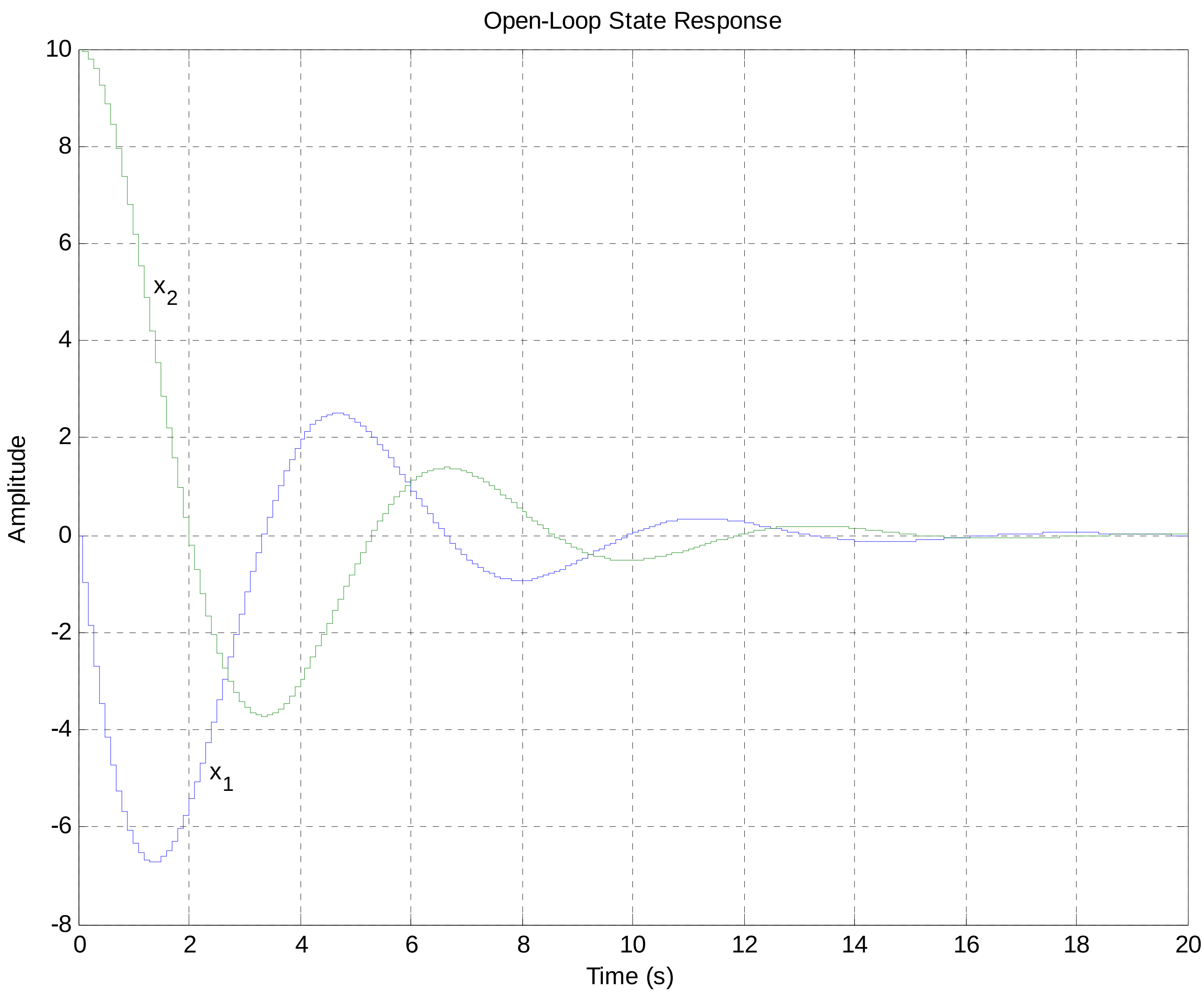
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.6 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Conversion to Discrete Time

$T = 0.1\,s\,,\qquad \text{zero-order hold}$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.9370 & -0.0969 \\ 0.0969 & 0.9951 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0969 \\ 0.0049 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$



Open-Loop State Response
Amplitude
Time (s)
10
8
6
4
2
0
-2
-4
-6
-8
0
2
4
6
8
10
12
14
16
18
20
x2
x1

Open-Loop Response

- Open-loop system is asymptotically stable, so both states decay to 0.
- States settle to within ±0.05 in 17.4 seconds.
- No control is used in this open-loop configuration.

Closed-Loop Pole Locations

for each $k \in [1, \ \ 10]$

$$\zeta = 0.7, \qquad \omega_n = k, \qquad T = 0.1\, s$$

$$\lambda_{cont\_k} = \omega_n \left[-\zeta \pm j\sqrt{1-\zeta^2} \right]$$

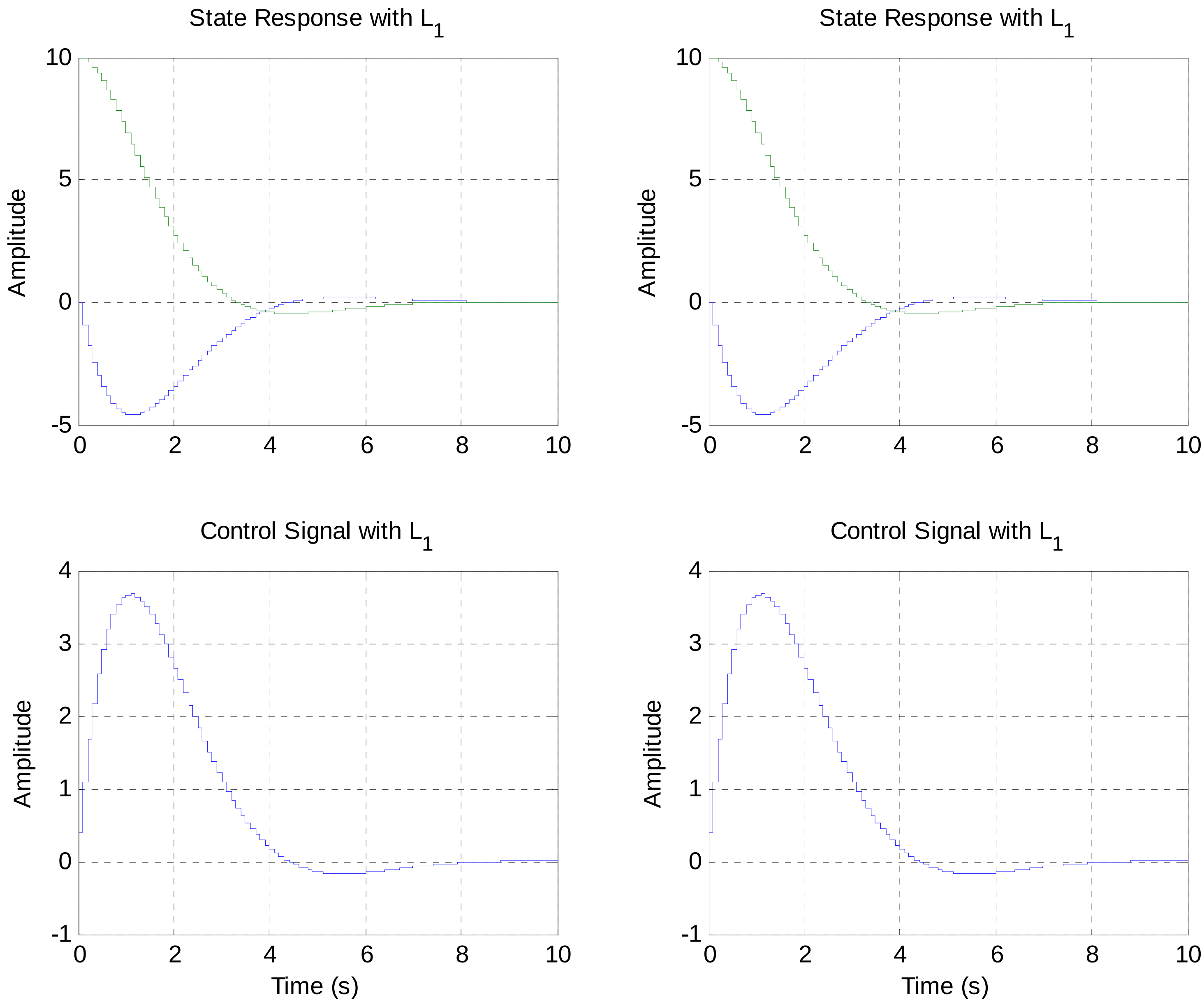
$$\lambda_{disc\_k} = e^{\lambda_{cont} T}$$

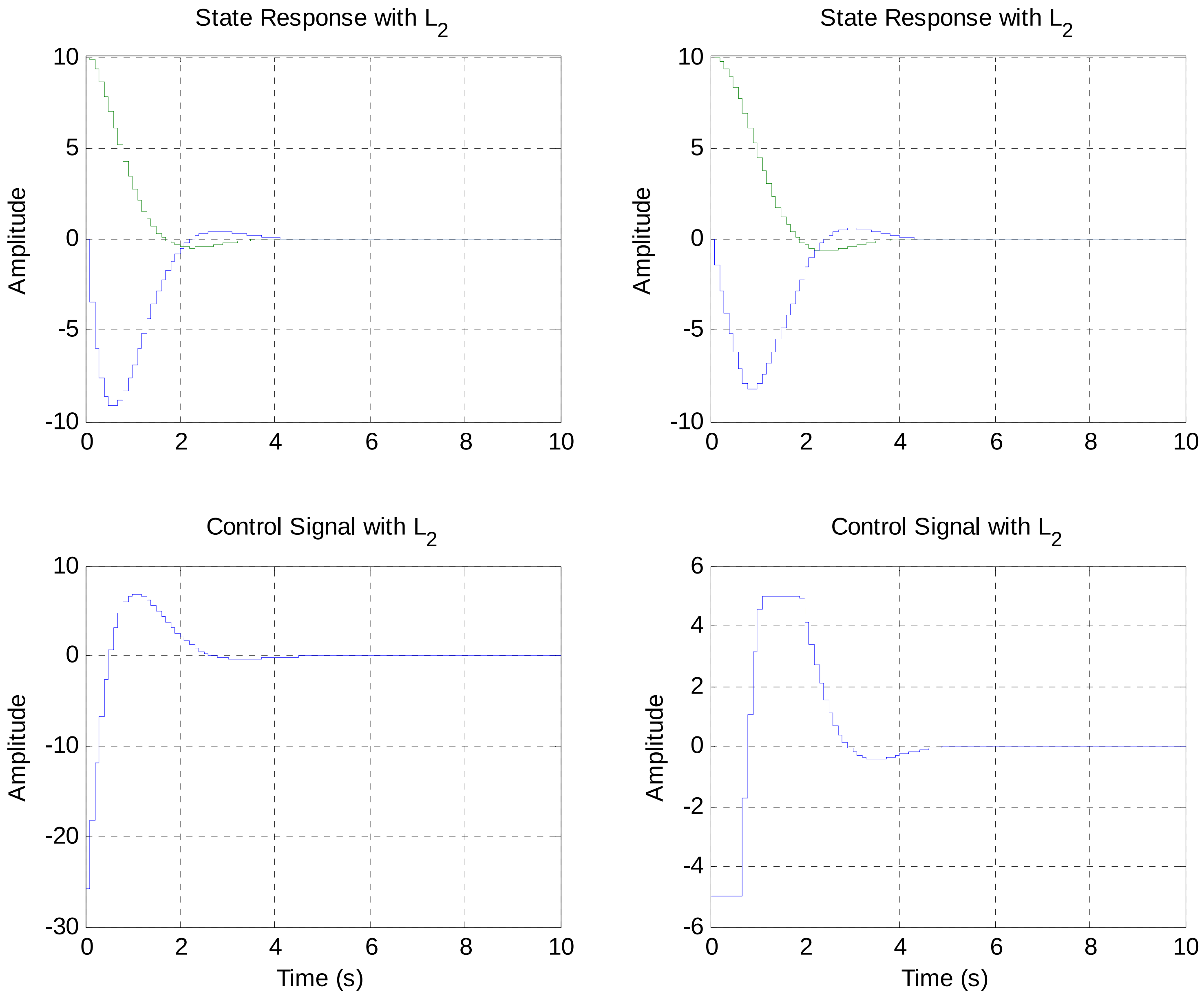
$$\lambda_{disc\_11} = \{0, 0\}$$

As k increases, closed-loop poles move farther away from open-loop poles and closer to the origin.

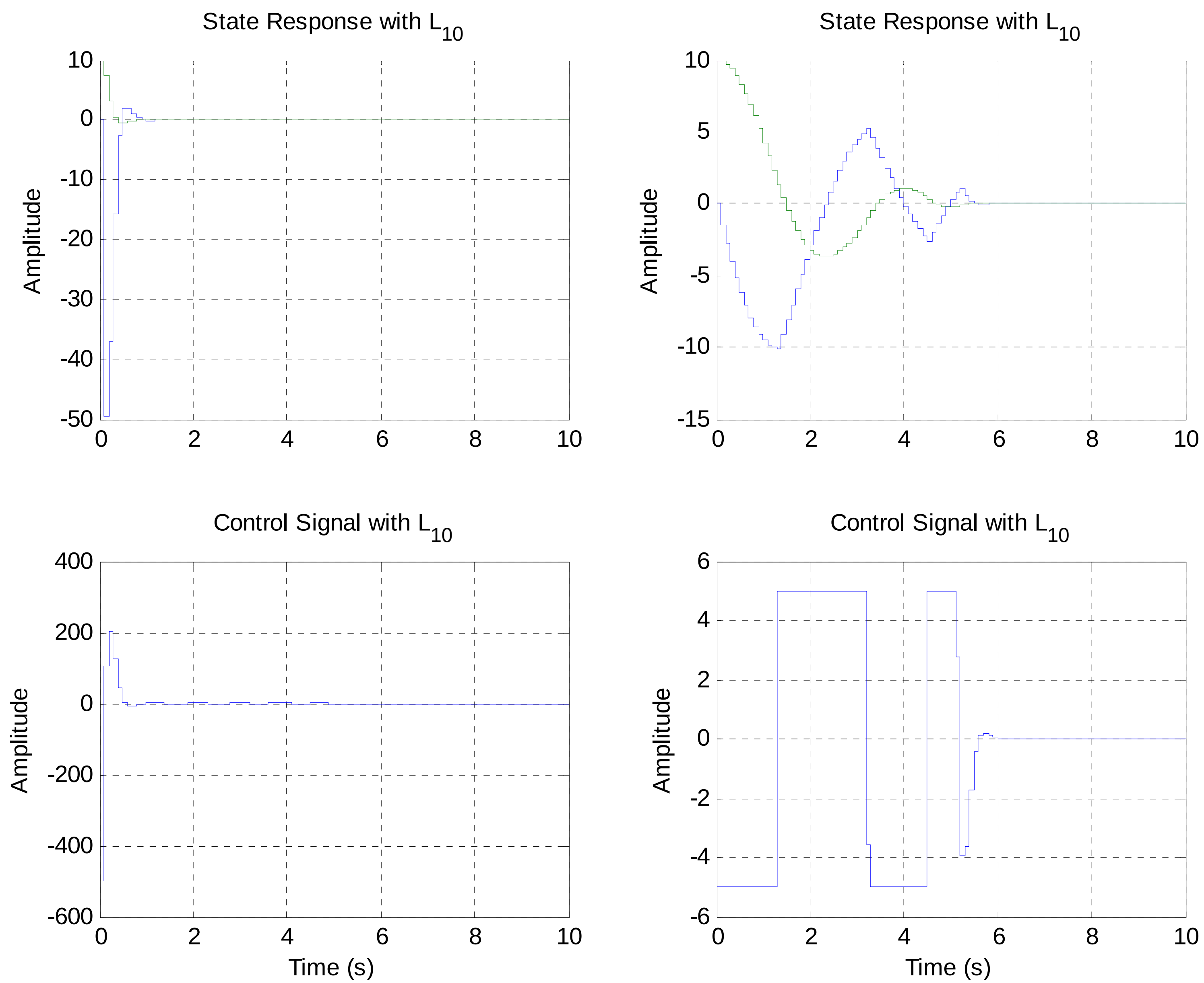
Closed-Loop Experiments

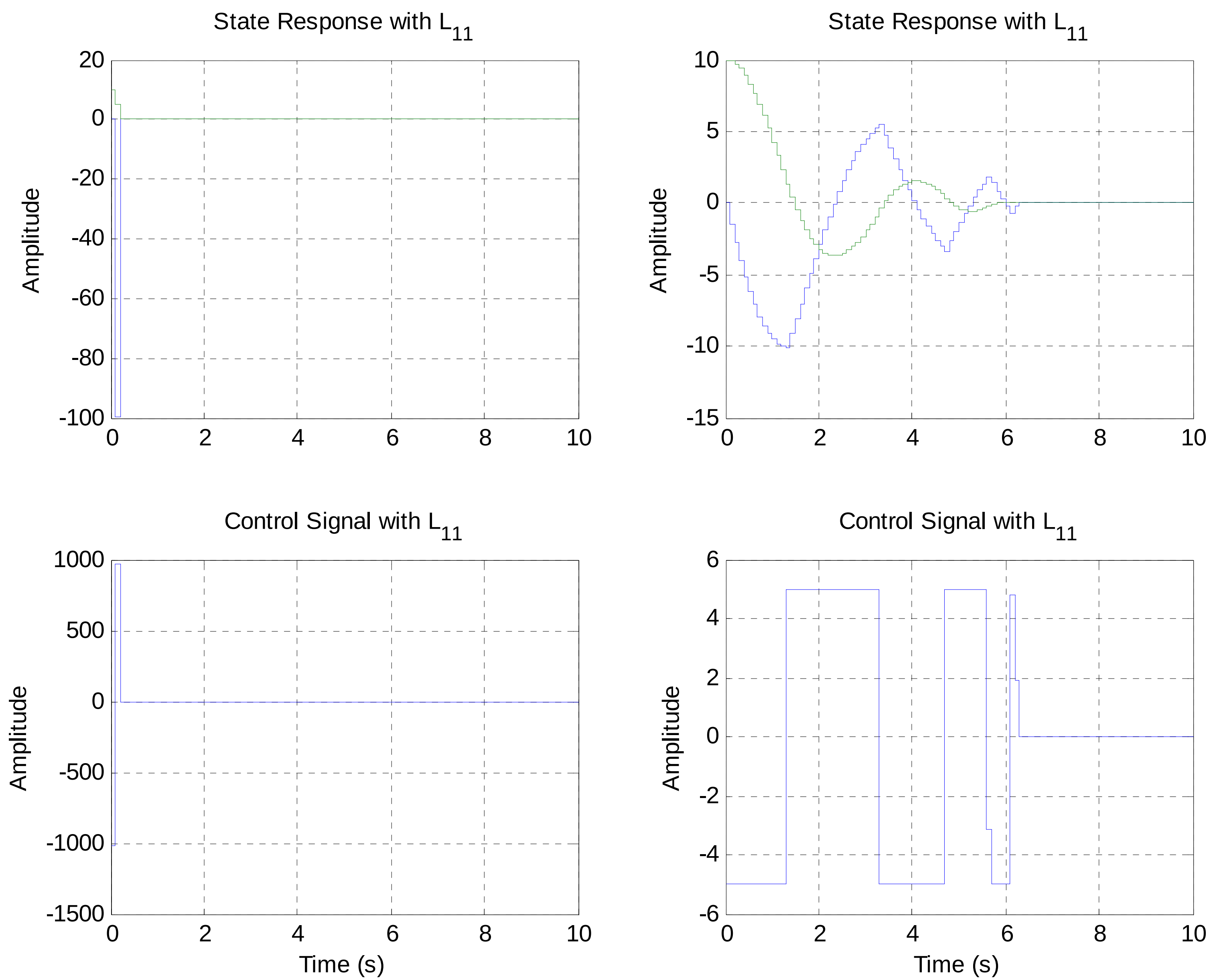
- For each set of closed-loop poles, the gain is computed, and closed-loop system is simulated with no control saturation.
- For the same set of closed-loop poles and gains, the closed-loop system is simulated with control saturation at ± 5.
- State responses and various performance measures are computed and plotted.


State Response with L_1
State Response with L_1
Amplitude
10
5
0
-5
0 2 4 6 8 10
Control Signal with L_1
Control Signal with L_1
Amplitude
4
3
2
1
0
-1
Time (s)



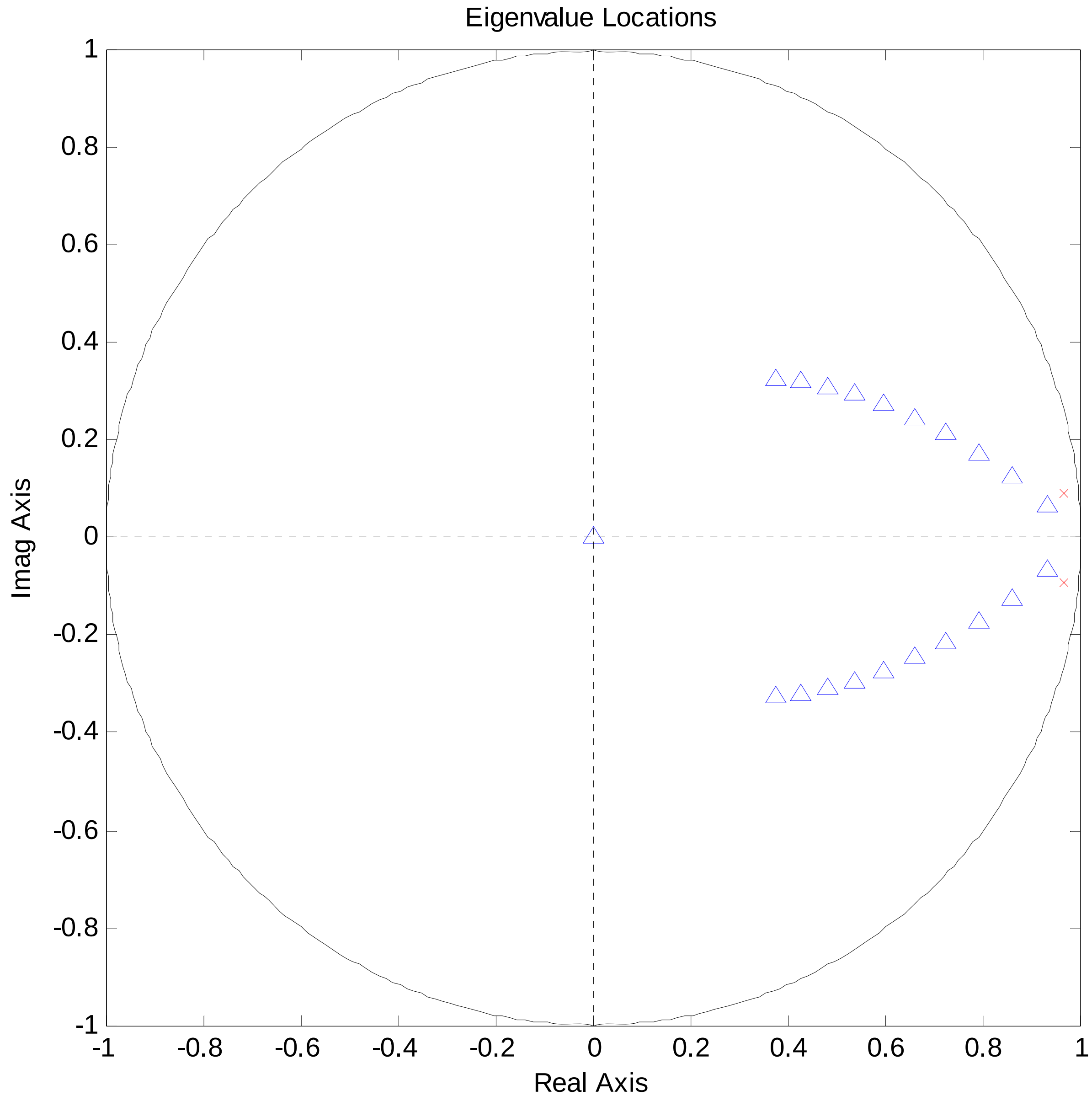
State Response with L2
Amplitude
Time (s)
State Response with L2
Amplitude
Control Signal with L2
Amplitude
Time (s)
Control Signal with L2
Amplitude
Time (s)


State Response with L_{10}
Amplitude
10
0
-10
-20
-30
-40
-50
0
2
4
6
8
10
State Response with L_{10}
Amplitude
10
5
0
-5
-10
-15
Control Signal with L_{10}
Amplitude
400
200
0
-200
-400
-600
Time (s)
Control Signal with L_{10}
Amplitude
6
4
2
0
-2
-4
-6
Time (s)

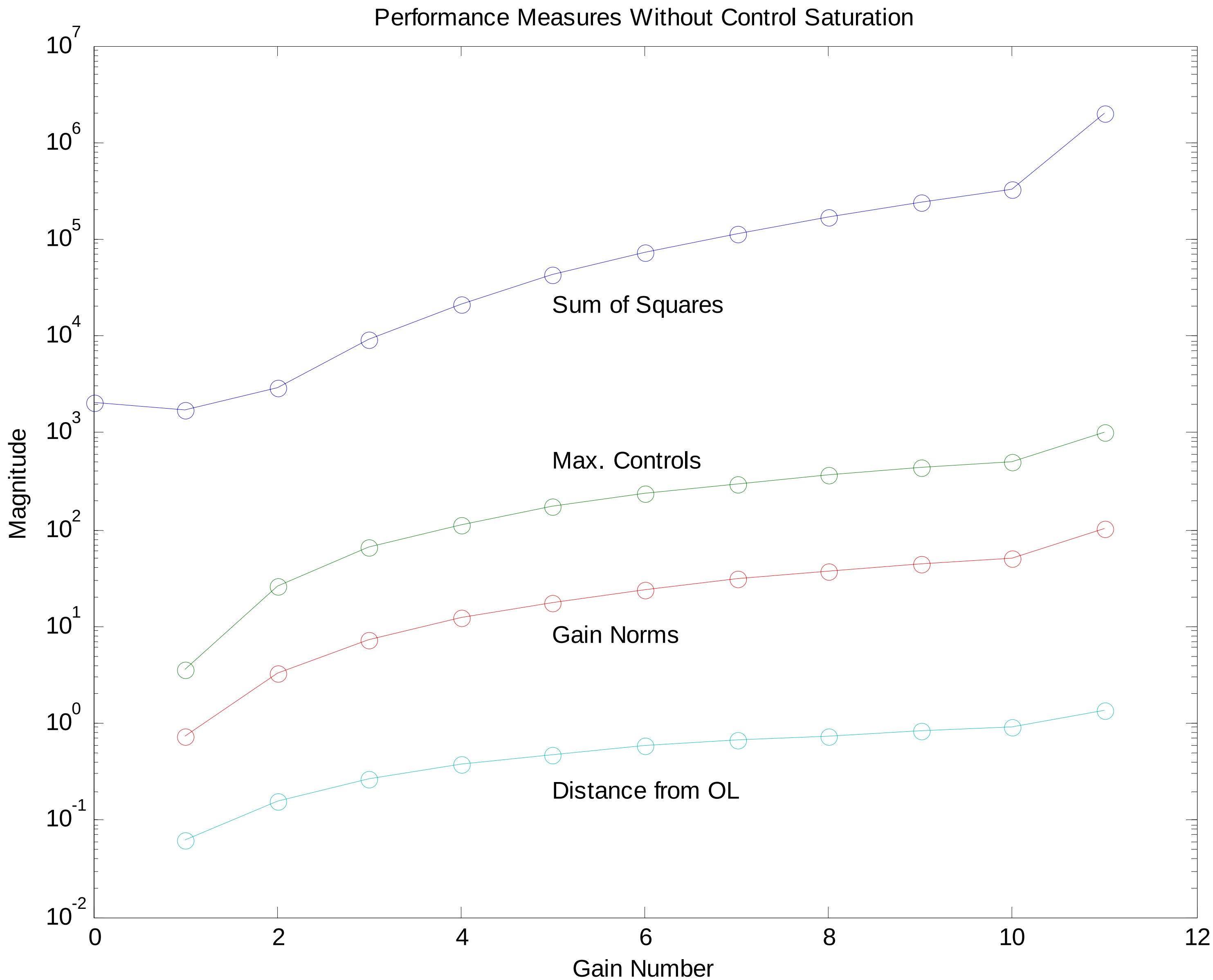

State Response with L_{11}
Amplitude
State Response with L_{11}
Amplitude
Control Signal with L_{11}
Amplitude
Time (s)
Control Signal with L_{11}
Amplitude
Time (s)

Closed-Loop Responses

- As natural frequency increases:
  - settling time decreases;
  - control signals become larger in magnitude;
  - state variable values are larger in magnitude.
- If control saturation is present:
  - amount of time in saturation increases with gain number;
  - settling time increases with gain number;
  - closed-loop stability is maintained.

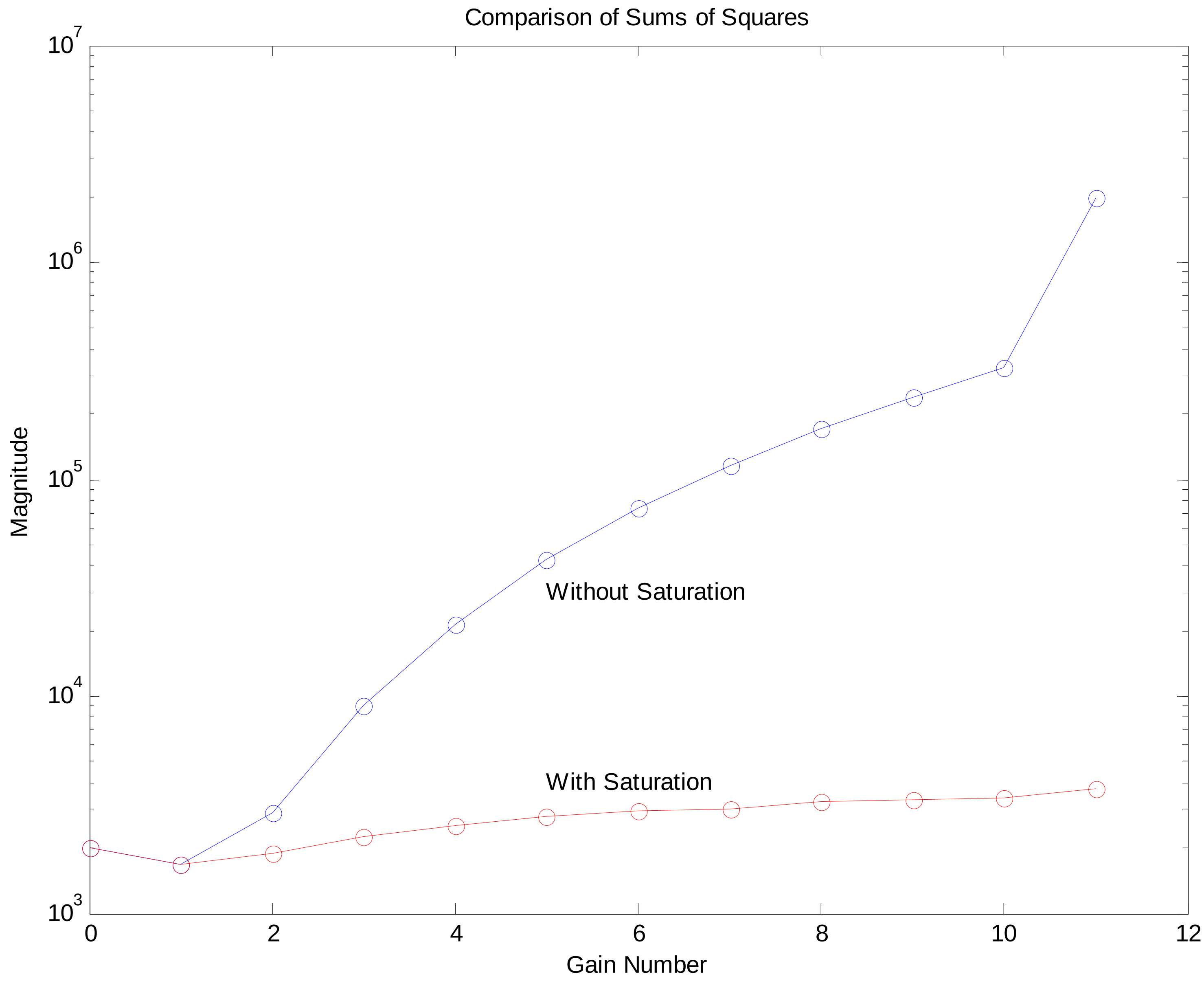


Eigenvalue Locations
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
Imag Axis
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Real Axis


Performance Measures Without Control Saturation
Magnitude
Gain Number
Sum of Squares
Max. Controls
Gain Norms
Distance from OL
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
0
2
4
6
8
10
12

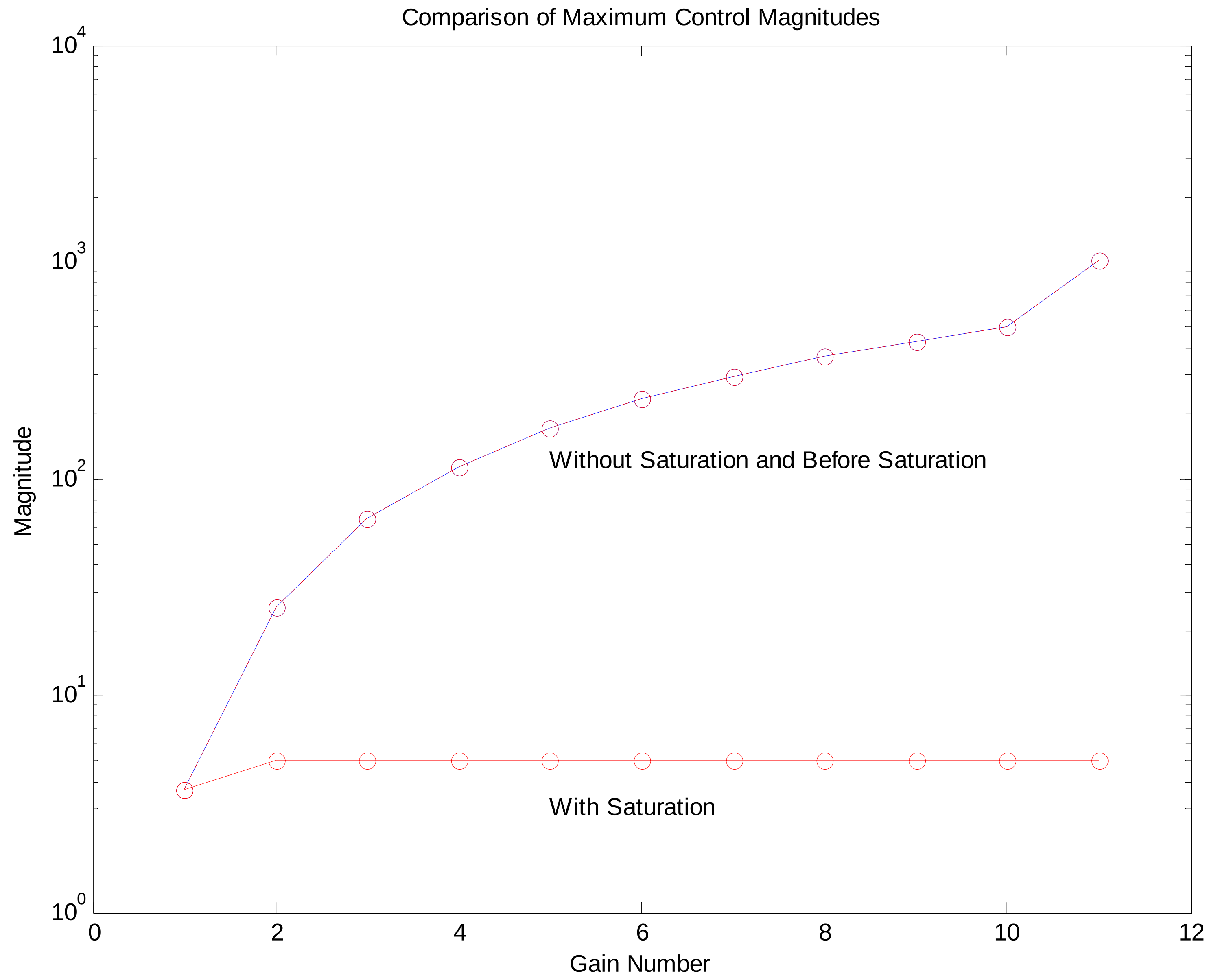
Performance Measures

- As distance of closed-loop poles from open-loop poles increases:
  - elements in gain matrix increase in magnitude;
  - control signal increases in magnitude because of the larger gain and larger state values;
  - state variables increase in magnitude because of the larger control signals;
  - the sum of squares increases because of the larger state and control magnitudes.


Comparison of Sums of Squares
Magnitude
10^7
10^6
10^5
10^4
10^3
0
2
4
6
8
10
12
Gain Number
Without Saturation
With Saturation

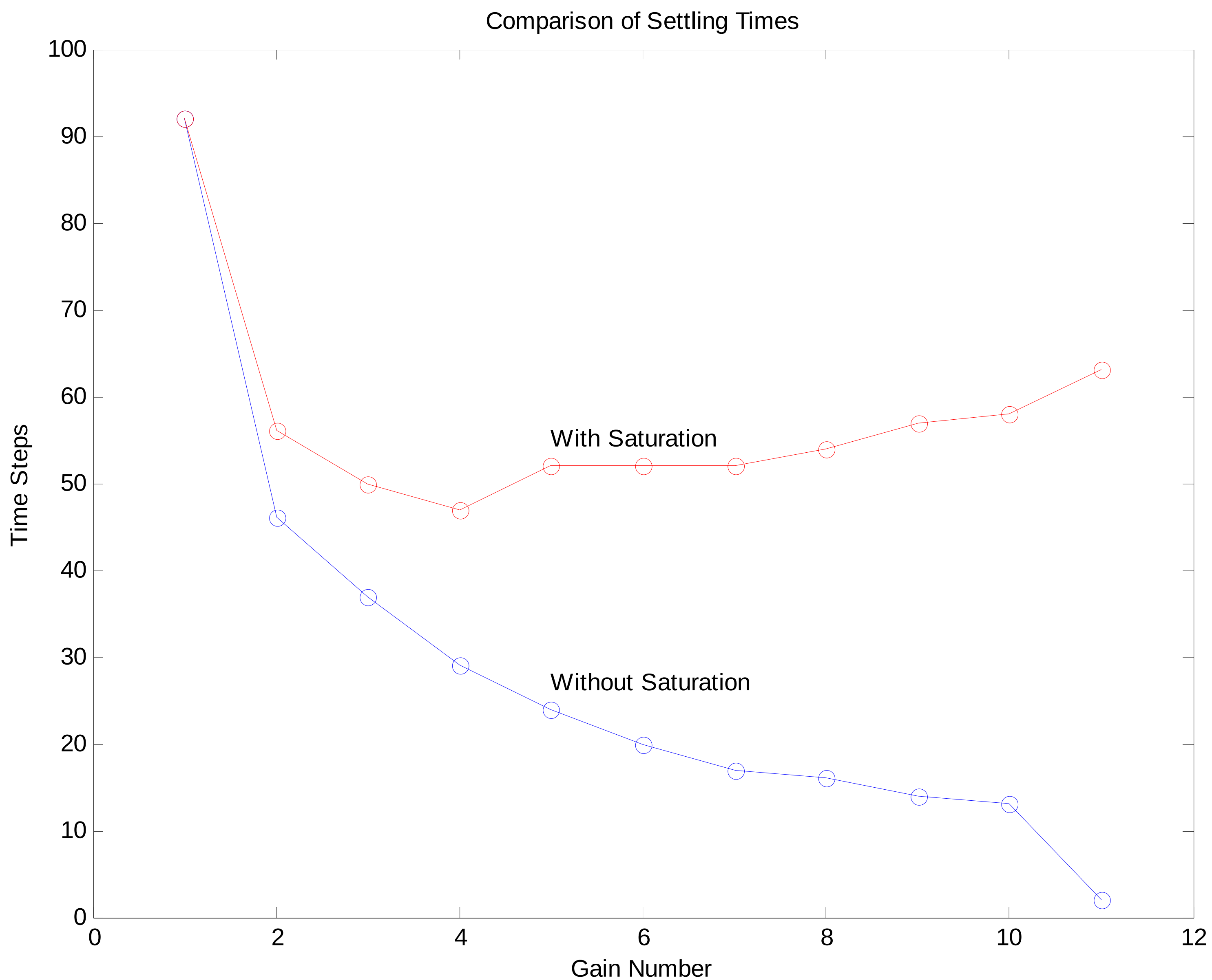
Sum of Squares

- $J = \sum(x^T x + u^T u)$
- With control saturation:
  - control signal is limited to ± 5, so it is smaller than without saturation for most gains;
  - smaller control signal produces smaller values for state variable magnitudes;
  - J is smaller with saturation than without, except for Gain Number 1 that does not saturate.


Comparison of Maximum Control Magnitudes
Magnitude
Gain Number
Without Saturation and Before Saturation
With Saturation
10^4
10^3
10^2
10^1
10^0
0
2
4
6
8
10
12

Maximum Control Magnitudes

- Control signals computed by the gain matrix have the same maximum magnitudes with and without control saturation.
  - The maximum magnitude control occurs at the initial time for all Gain Numbers > 1.
  - With Gain Number 1, the peak magnitude is less than 5, so there is no saturation.
- For large Gain Numbers, the control signals before saturation have more oscillations and longer settling times.


Comparison of Settling Times
Time Steps
Gain Number
With Saturation
Without Saturation
0
10
20
30
40
50
60
70
80
90
100
0
2
4
6
8
10
12

Settling Times

- Without saturation, settling time decreases as the closed-loop eigenvalues move toward the origin.
- Without saturation, when the closed-loop eigenvalues are at the origin, the settling time is 2 time steps, the order of the system.
- With saturation, settling time increases rather than decreases with the faster eigenvalues due to larger control magnitudes.