Digital Control Systems
State Feedback Control

Illustrating the Effects of Closed-Loop Eigenvalue Location and Saturation for an Unstable Open-Loop System

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Continuous-Time System

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 - 0.6s + 1}, \qquad \zeta = -0.3, \omega_n = 1\, r/s$$

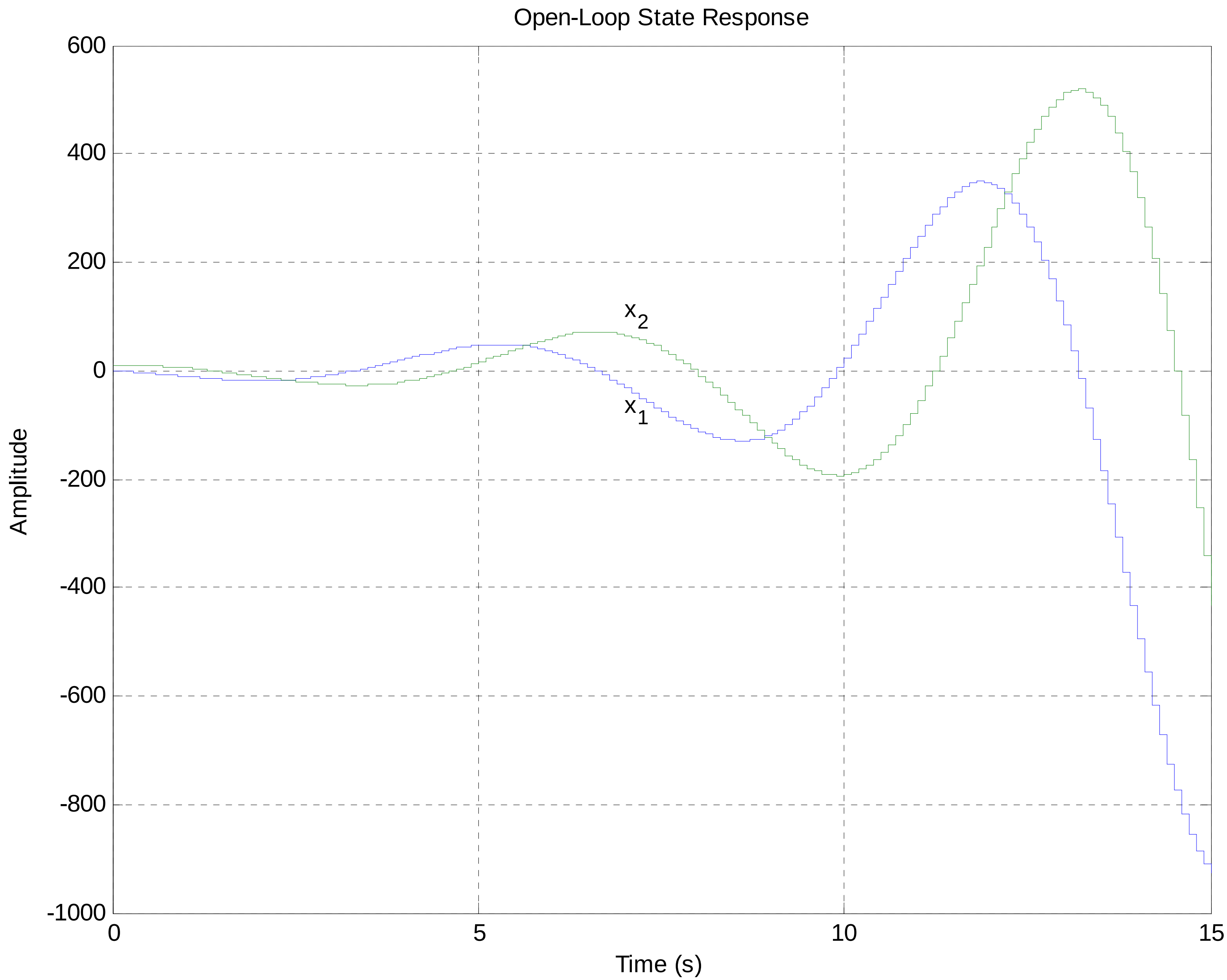
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0.6 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Conversion to Discrete Time

$T = 0.1\,s$, zero-order hold

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.0566 & -0.1029 \\ 0.1029 & 0.9949 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.1029 \\ 0.0051 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$



Open-Loop State Response
Amplitude
600
400
200
0
-200
-400
-600
-800
-1000
0
5
10
15
Time (s)
x_2
x_1

Open-Loop Response

- Open-loop system is unstable, so the state response grows without bound.
- Settling time and similar measures are undefined.
- No control is used in this open-loop configuration.

Closed-Loop Pole Locations

for each $k \in [1, \ 10]$

$$\zeta = 0.7, \quad \omega_n = k, \quad T = 0.1\, s$$

$$\lambda_{cont\_k} = \omega_n \left[-\zeta \pm j\sqrt{1-\zeta^2} \right]$$

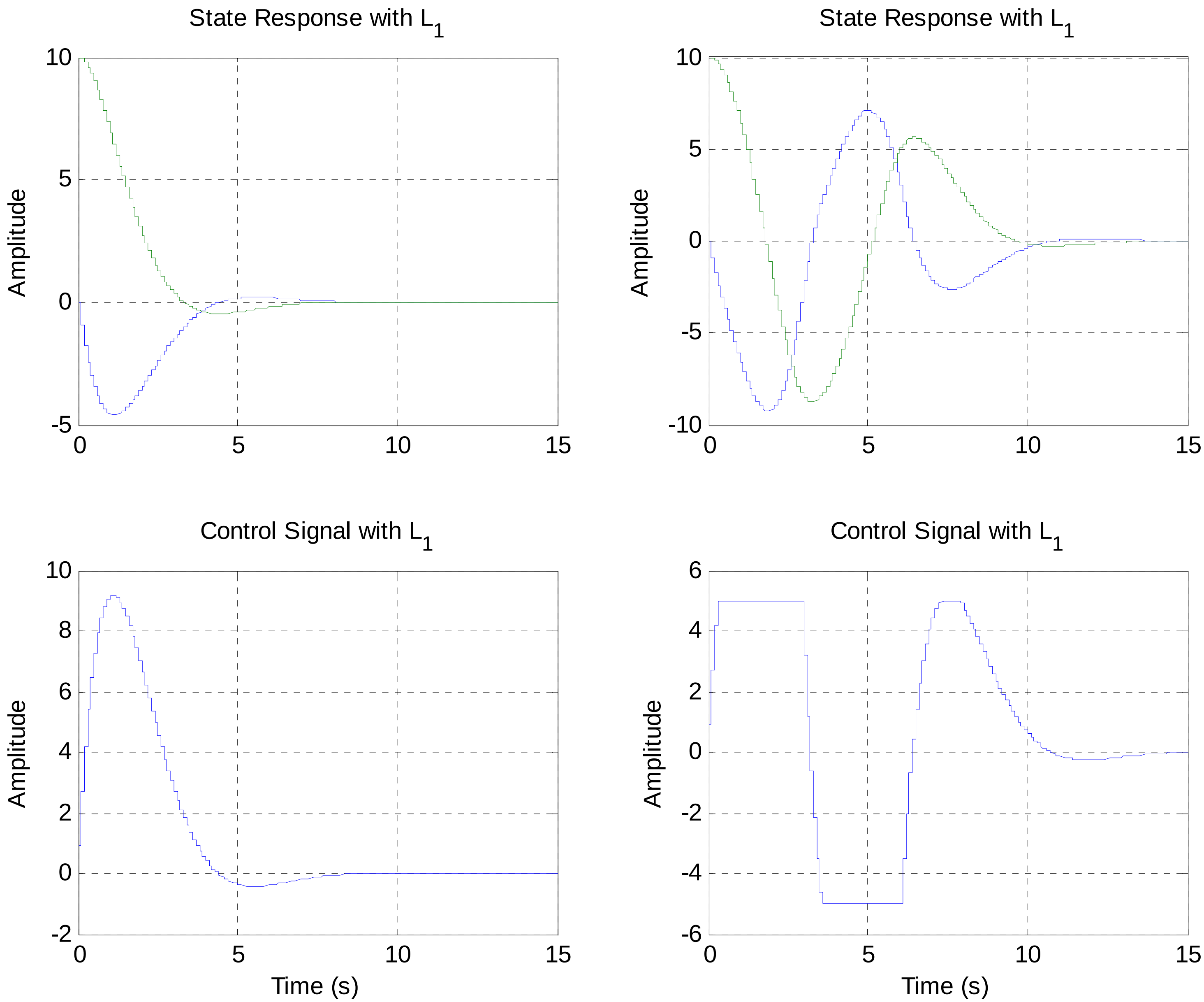
$$\lambda_{disc\_k} = e^{\lambda_{cont} T}$$

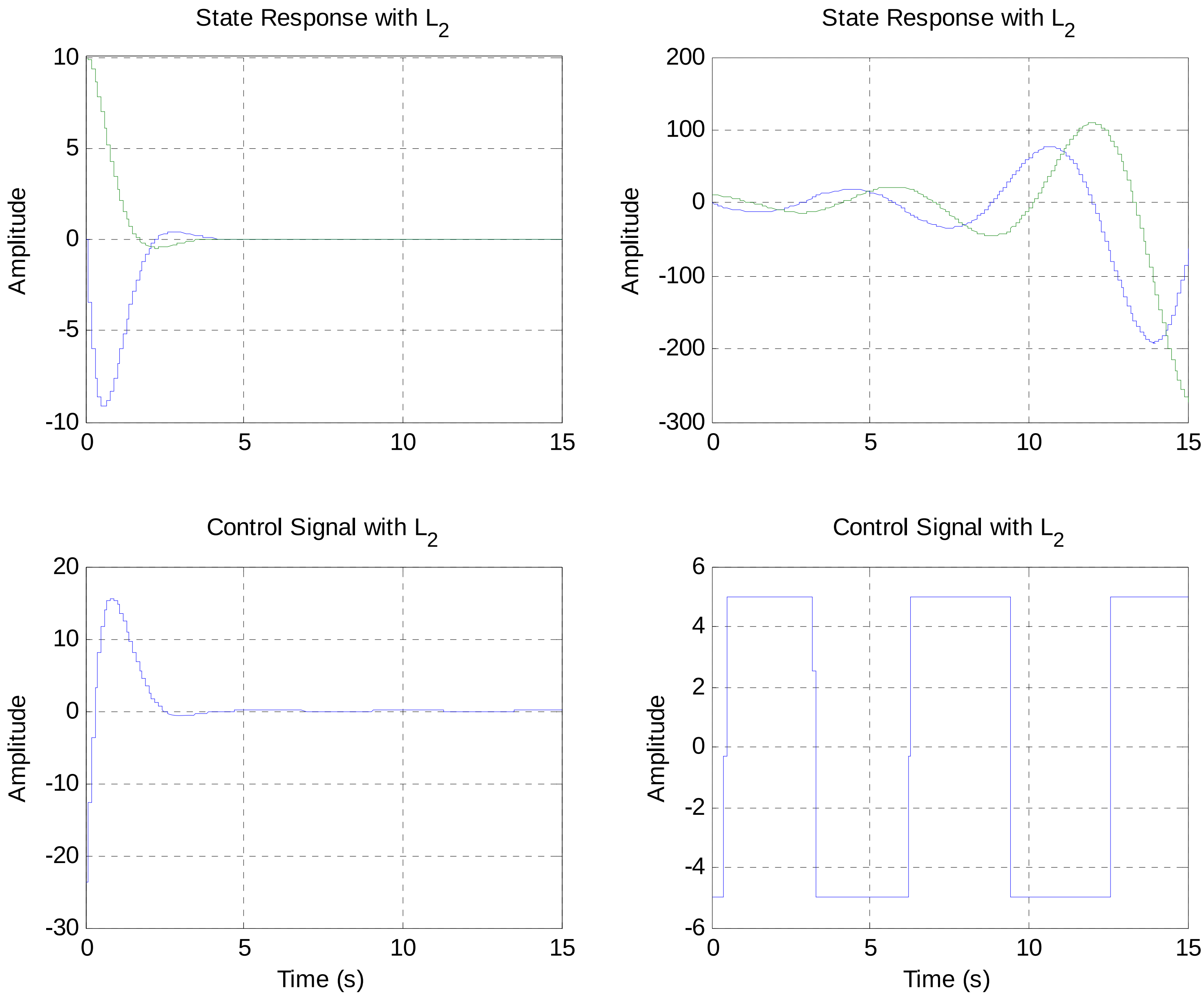
$$\lambda_{disc\_11} = \{0, 0\}$$

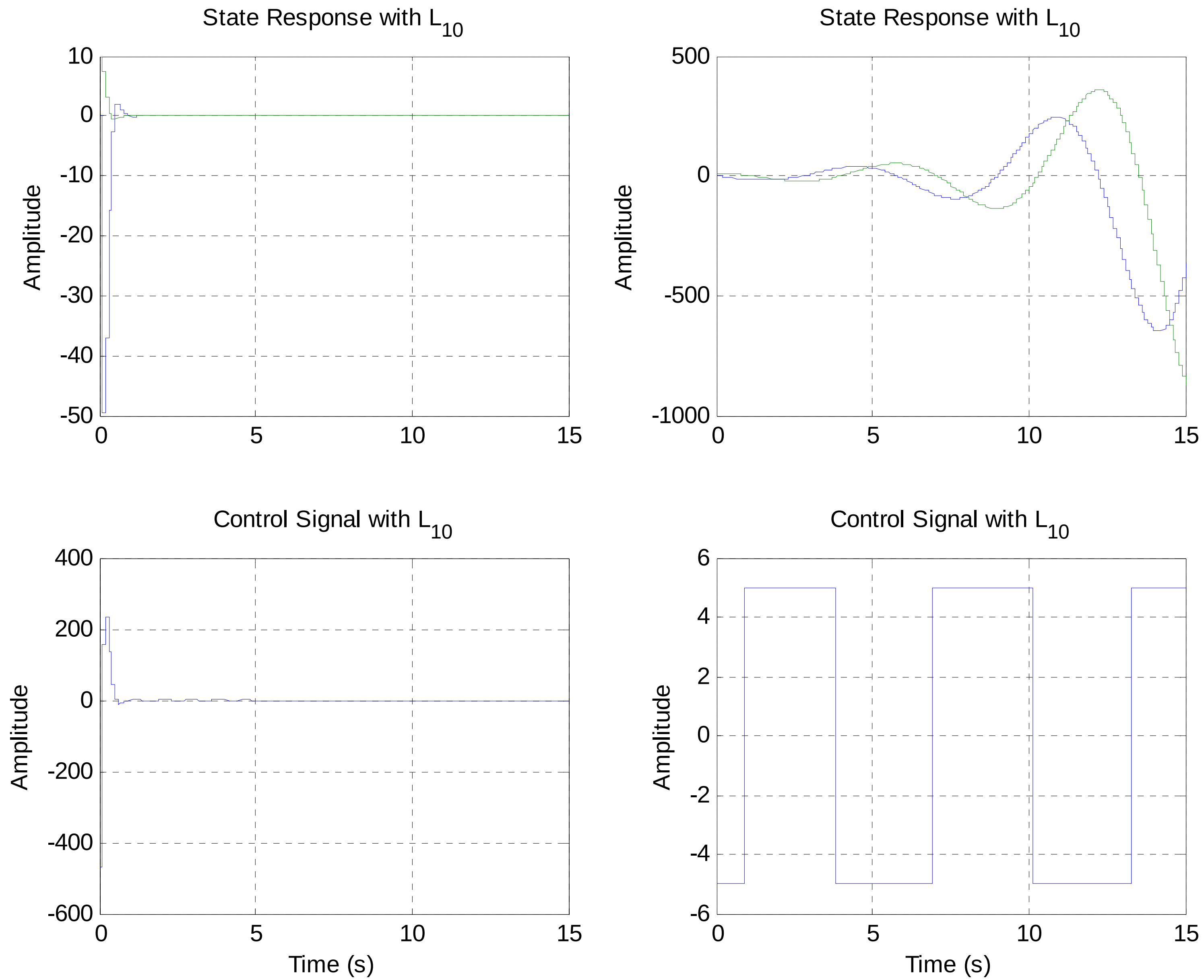
As k increases, closed-loop poles move farther away from open-loop poles and closer to the origin.

Closed-Loop Experiments

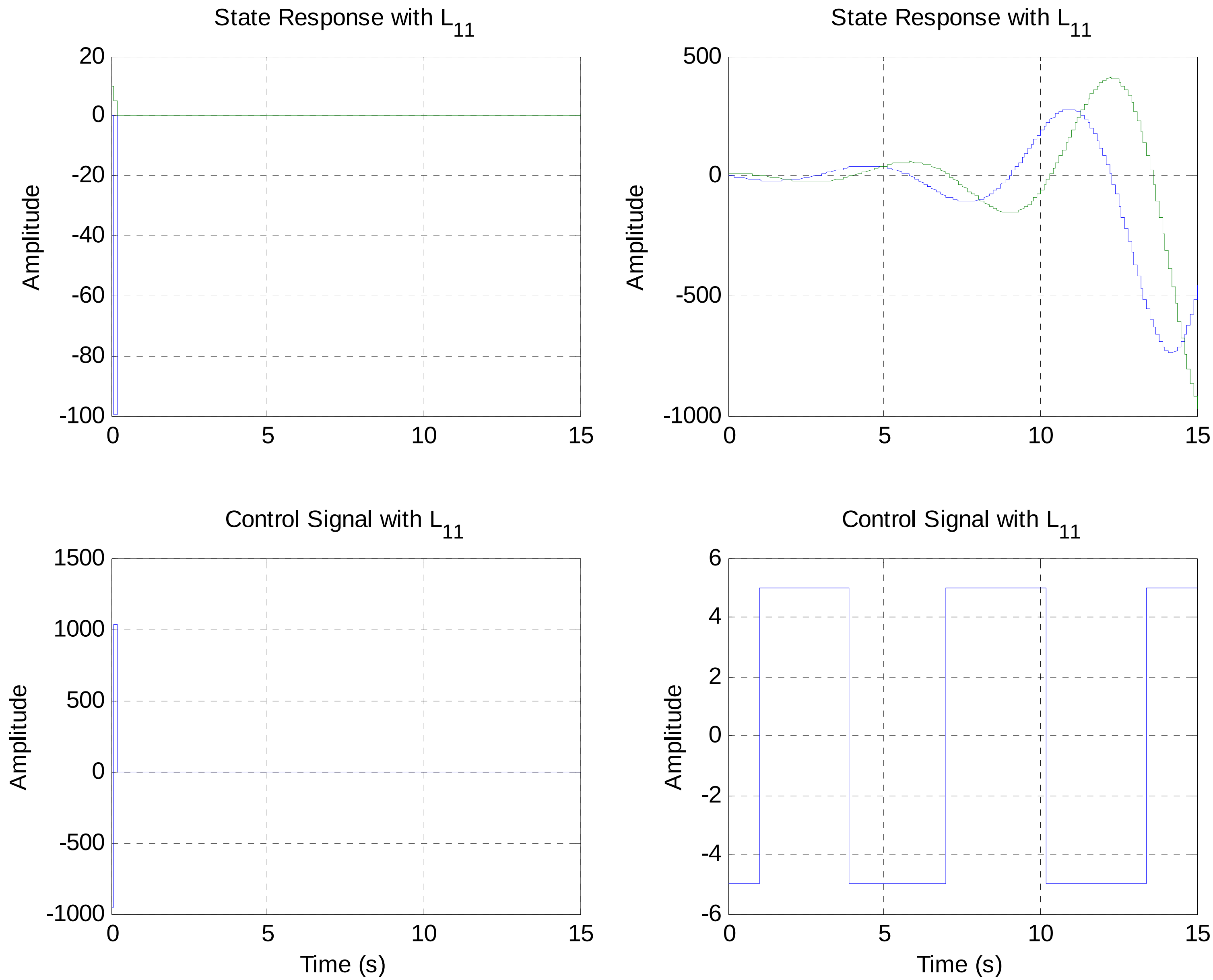
- For each set of closed-loop poles, the gain is computed, and closed-loop system is simulated with no control saturation.
- For the same set of closed-loop poles and gains, the closed-loop system is simulated with control saturation at ± 5.
- State responses and various performance measures are computed and plotted.


State Response with L_1
Amplitude
Control Signal with L_1
Time (s)
10
5
0
-5
-10
8
6
4
2
-2
-4
-6
15


State Response with L_2
Amplitude
State Response with L_2
Amplitude
Control Signal with L_2
Amplitude
Time (s)
Control Signal with L_2
Amplitude
Time (s)



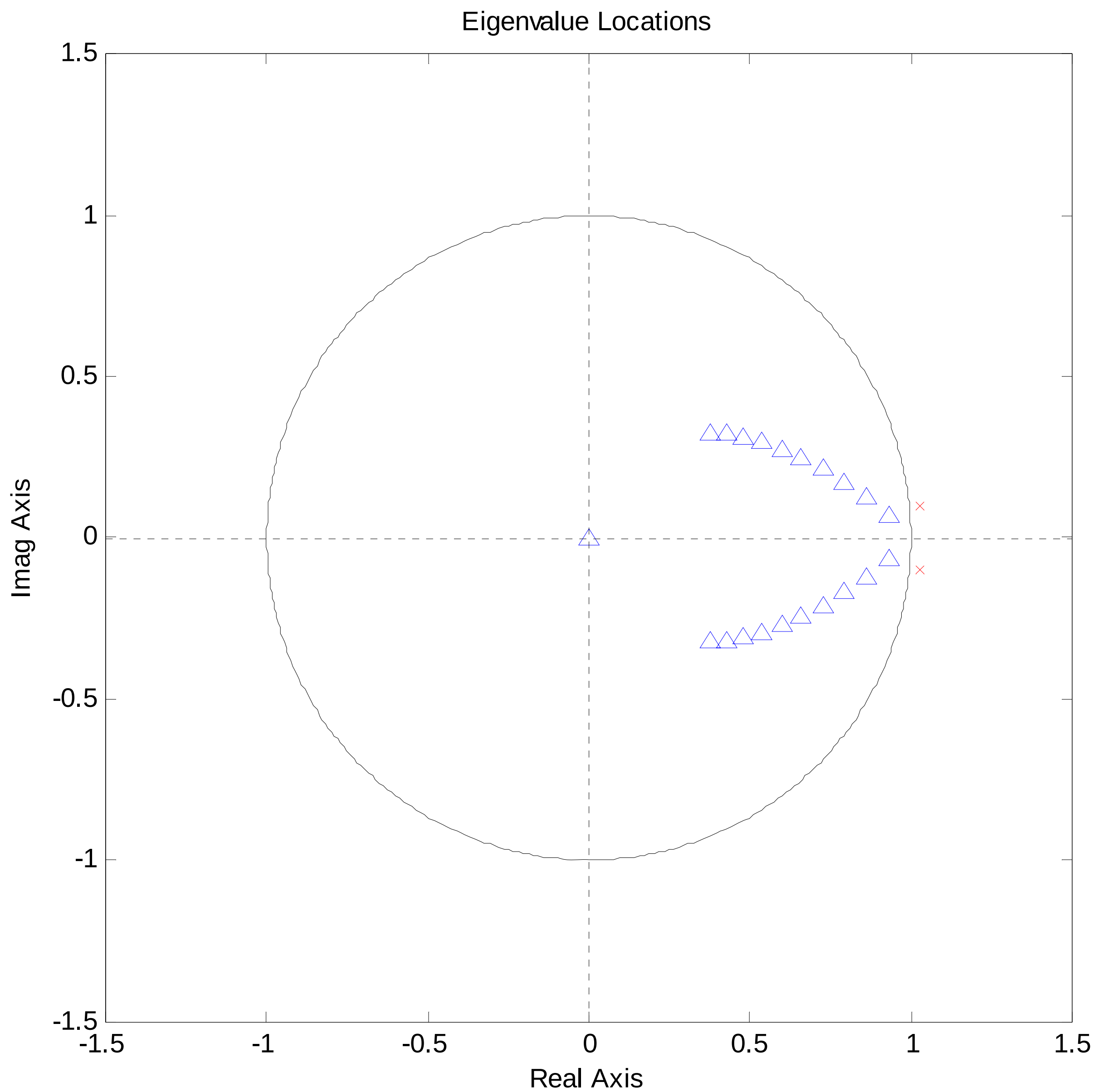
State Response with L10
Amplitude
10
0
-10
-20
-30
-40
-50
0
5
10
15
State Response with L10
Amplitude
500
0
-500
-1000
0
5
10
15
Control Signal with L10
Amplitude
400
200
0
-200
-400
-600
0
5
10
15
Time (s)
Control Signal with L10
Amplitude
6
4
2
0
-2
-4
-6
0
5
10
15
Time (s)



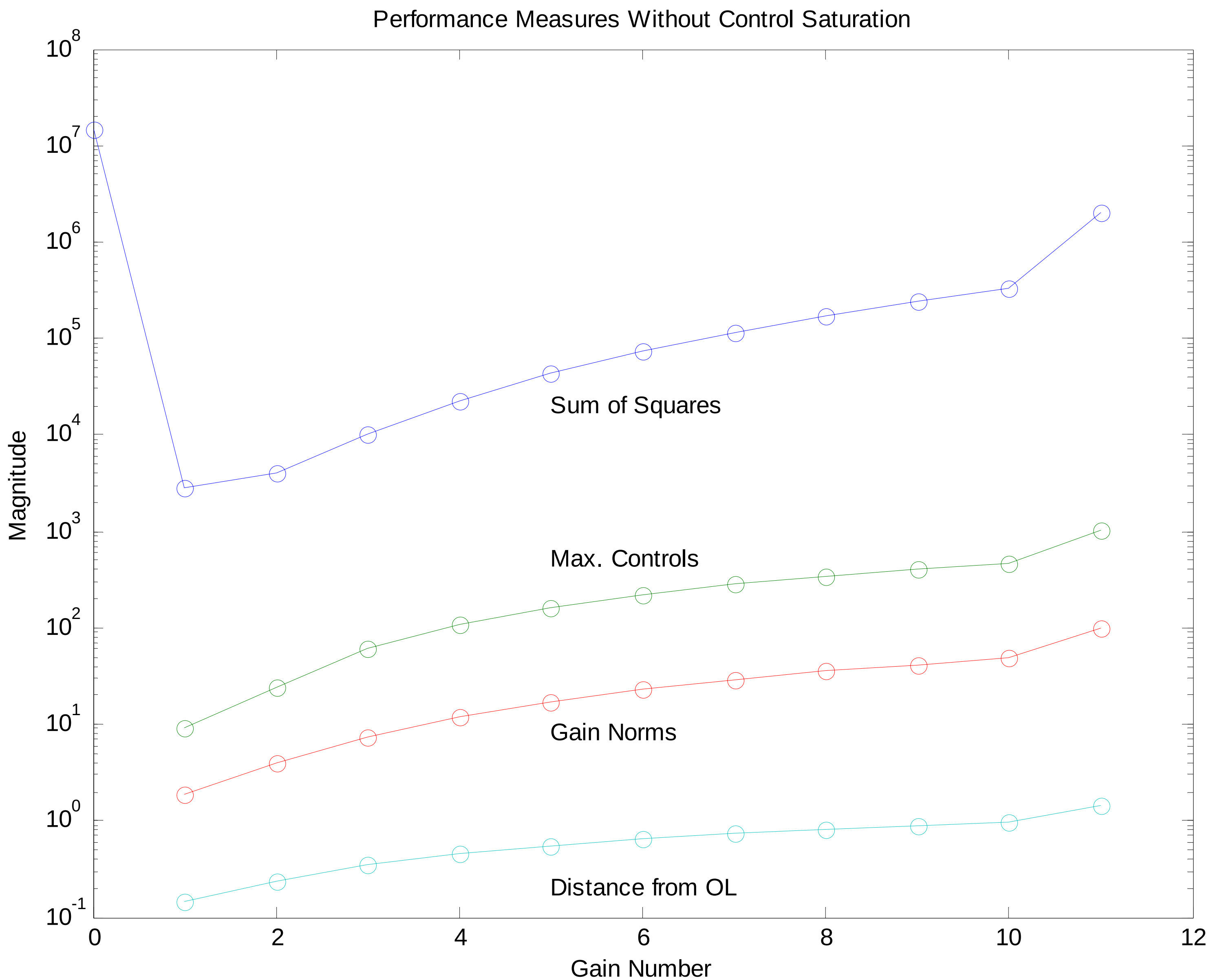
State Response with L11
State Response with L11
Control Signal with L11
Control Signal with L11
Amplitude
Time (s)
20
0
-20
-40
-60
-80
-100
500
-500
-1000
1500
1000
6
4
2
-2
-4
-6
5
10
15

Closed-Loop Responses

- Without control saturation:
  - control values and state variables increase in magnitude with increasing natural frequency;
  - settling time decreases.
- If control saturation is present:
  - all the control signals saturate;
  - Gain Number 1 is able to maintain stability;
  - the larger Gain Numbers are unable to maintain stability.



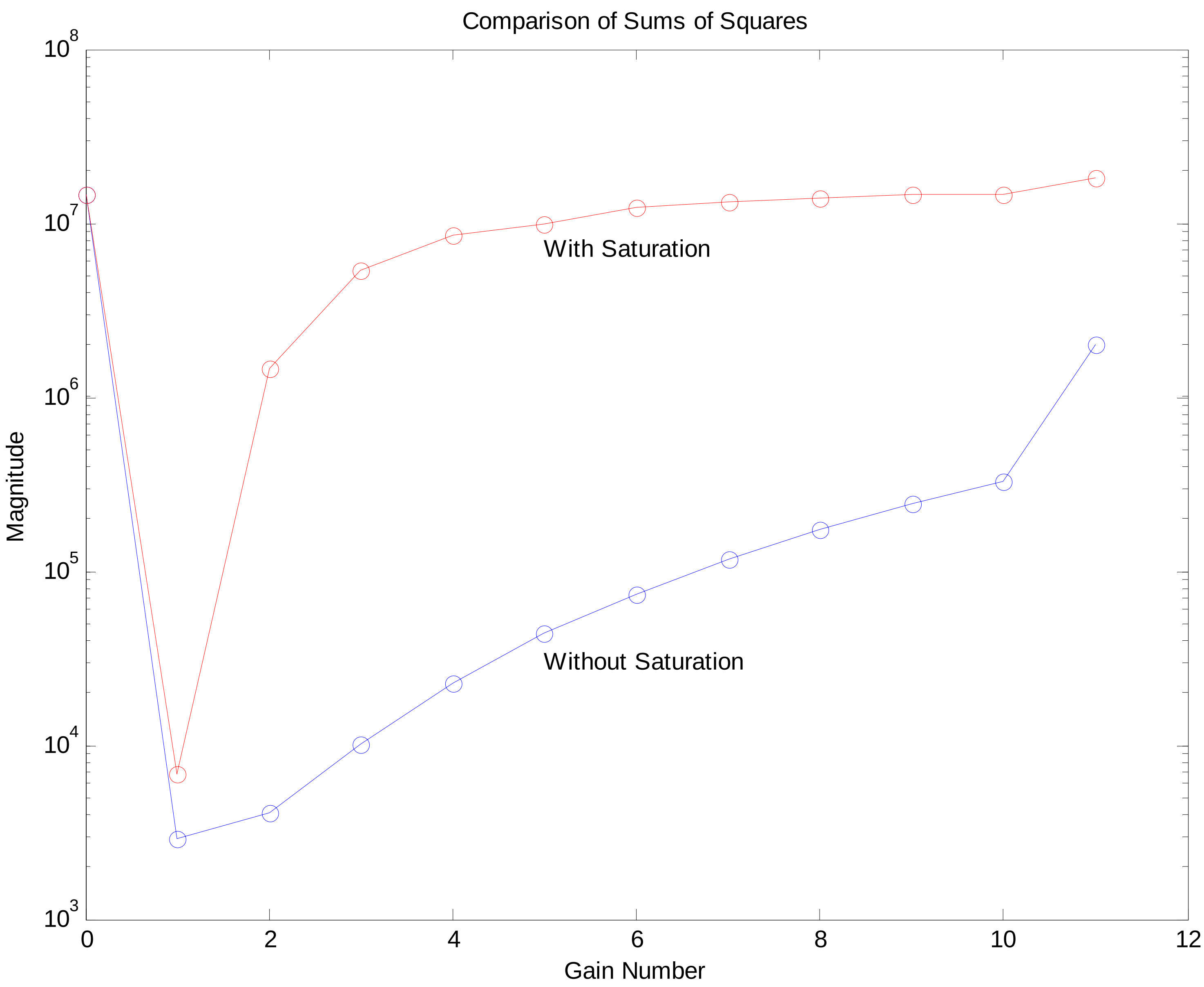
Eigenvalue Locations
1.5
1
0.5
0
-0.5
-1
-1.5
Imag Axis
-1.5
-1
-0.5
0
0.5
1
1.5
Real Axis



Performance Measures Without Control Saturation
Magnitude
Gain Number
Sum of Squares
Max. Controls
Gain Norms
Distance from OL
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
0
2
4
6
8
10
12

Performance Measures

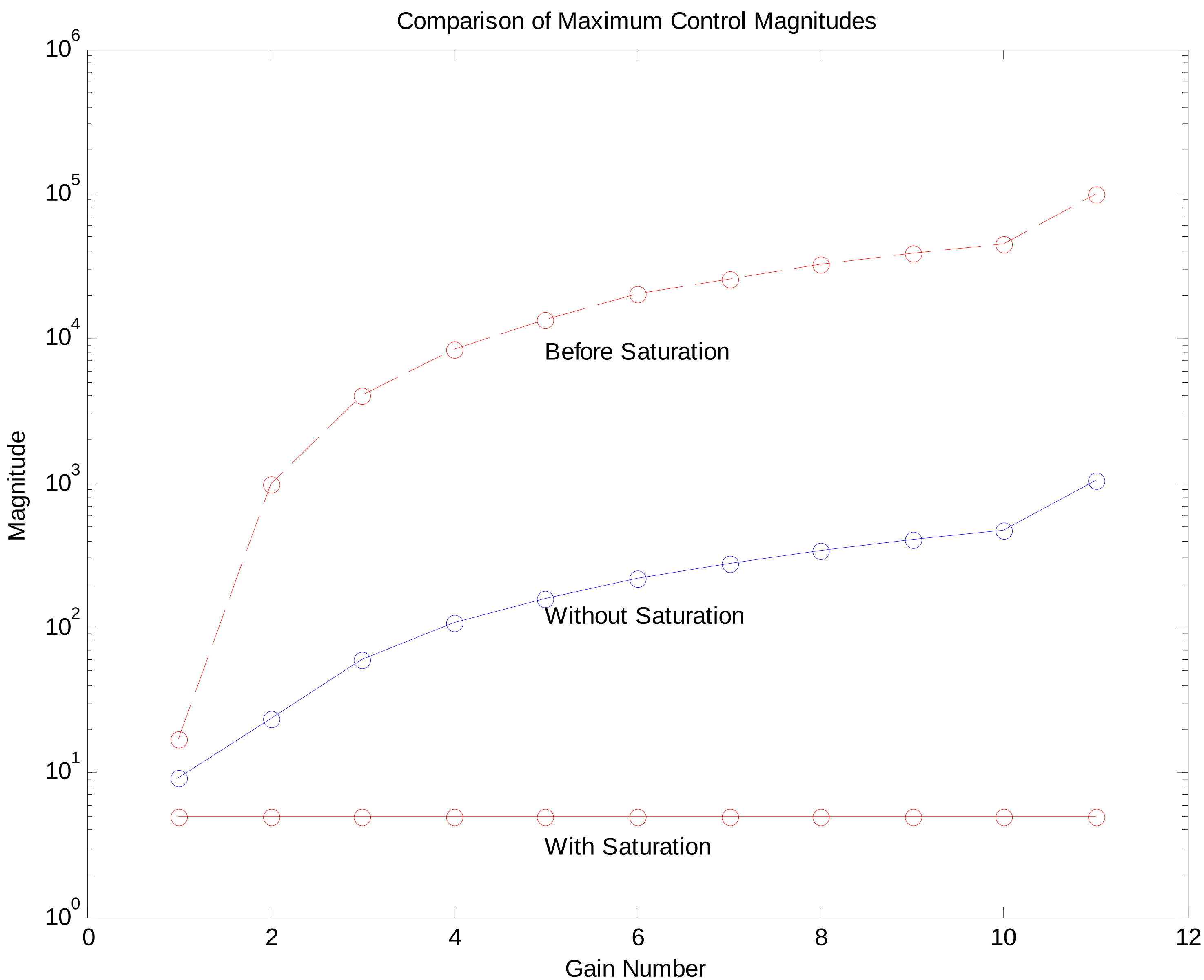
- As distance of closed-loop poles from open-loop poles increases:
  - elements in gain matrix increase in magnitude;
  - magnitudes of control signal and state variables increase in value;
  - the sum of squares increases because of the larger state and control magnitudes.
- A large decrease in sum of squares is observed when the loop is closed.



Comparison of Sums of Squares
10^8
10^7
10^6
10^5
10^4
10^3
Magnitude
With Saturation
Without Saturation
0
2
4
6
8
10
12
Gain Number

Sum of Squares

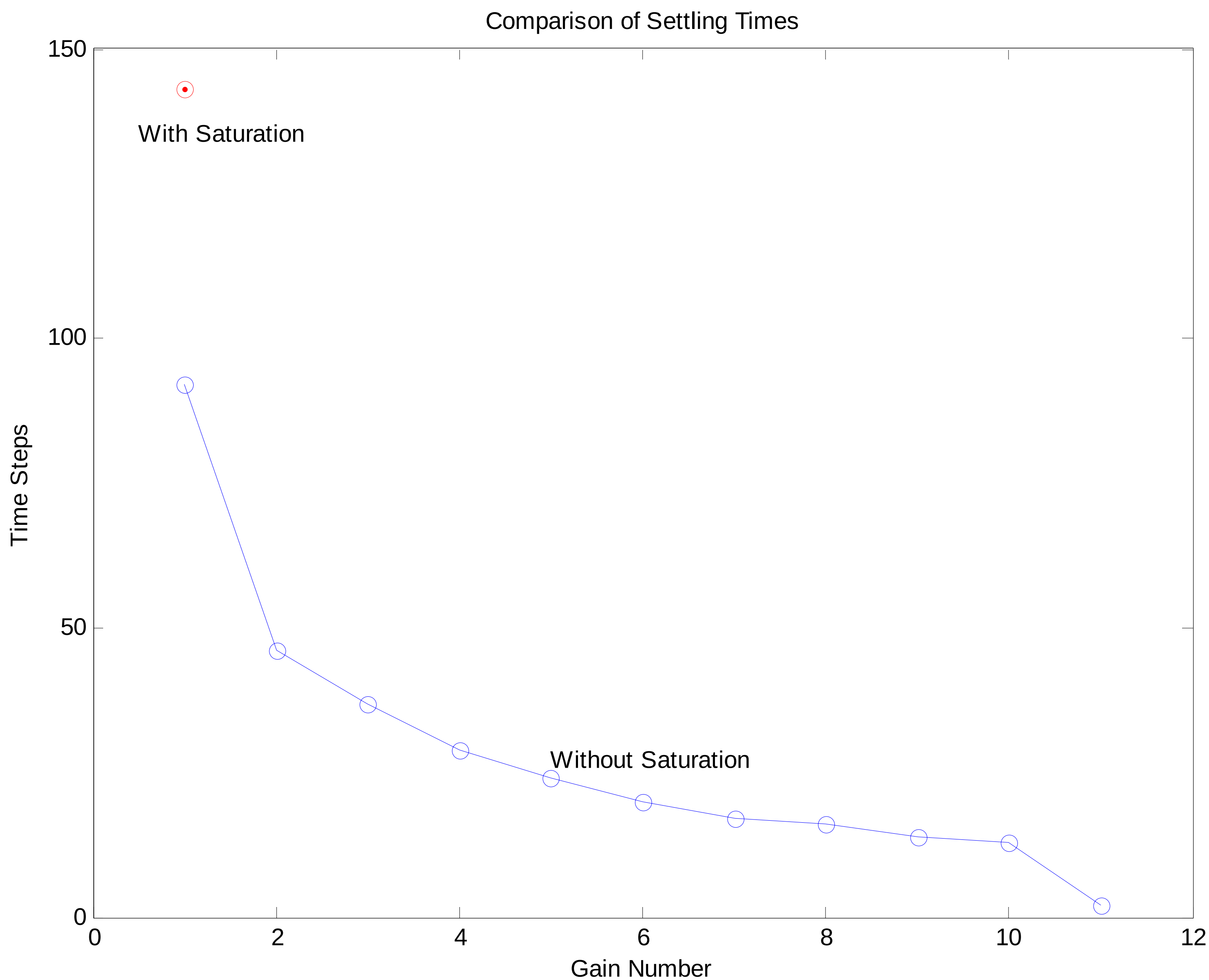
- $J = \sum(x^T x + u^T u)$
- With control saturation:
  - control signal is limited to ±5, so it is smaller than without saturation for all gains;
  - open-loop situation produced by controller saturation causes the states to grow in magnitude;
  - J will go to infinity as the simulation time is increased to infinity.



Comparison of Maximum Control Magnitudes
Magnitude
Gain Number
Before Saturation
Without Saturation
With Saturation
10^6
10^5
10^4
10^3
10^2
10^1
10^0
0
2
4
6
8
10
12

Maximum Control Magnitudes

- All control signals saturate.
- Control signals computed by the gain matrix when saturation is present are much larger than the control signals without controller saturation.
  - When the control signal is saturated, the state variables increase in magnitude.
  - The larger state variable magnitudes produce larger control signal magnitudes.


Comparison of Settling Times
150
100
50
0
Time Steps
With Saturation
Without Saturation
0
2
4
6
8
10
12
Gain Number

Settling Time

- Without saturation, settling time decreases as the closed-loop eigenvalues move toward the origin.
- Without saturation, when the closed-loop eigenvalues are at the origin, the settling time is 2 time steps, the order of the system.
- With saturation, closed-loop system is unstable except for Gain Number 1, so settling time is undefined.