Multivariable Control: Placement of Eigenvectors

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Presentation Outline

- Problem Statement
- Motivating Example
- Why Eigenvectors Are Important
- How Eigenvectors Can Be Found
- Which Eigenvectors Might Be Selected
- Example Continued
- Conclusions

Problem Statement

- Given either state-space description for a system:

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \qquad u(t) = -Lx(t)$$

$$x(k+1) = \Phi x(k) + \Gamma u(k)\,, \qquad u(k) = -Lx(k)$$

- Choose feedback gain L so that:
  - Eigenvalues are placed at specified locations;

$$\left|\lambda I - A + BL\right| = 0 \quad \text{or} \quad \left|\lambda I - \Phi + \Gamma L\right| = 0$$

  - Good performance is achieved.

Problem Statement

- For a single-input system, L is unique for a given set of eigenvalues.

- For a multi-input system, L is not unique for a given set of eigenvalues.

- For the same eigenvalues, different L matrices will produce:
  - Different eigenvectors;
  - Different performance.

Motivating Example

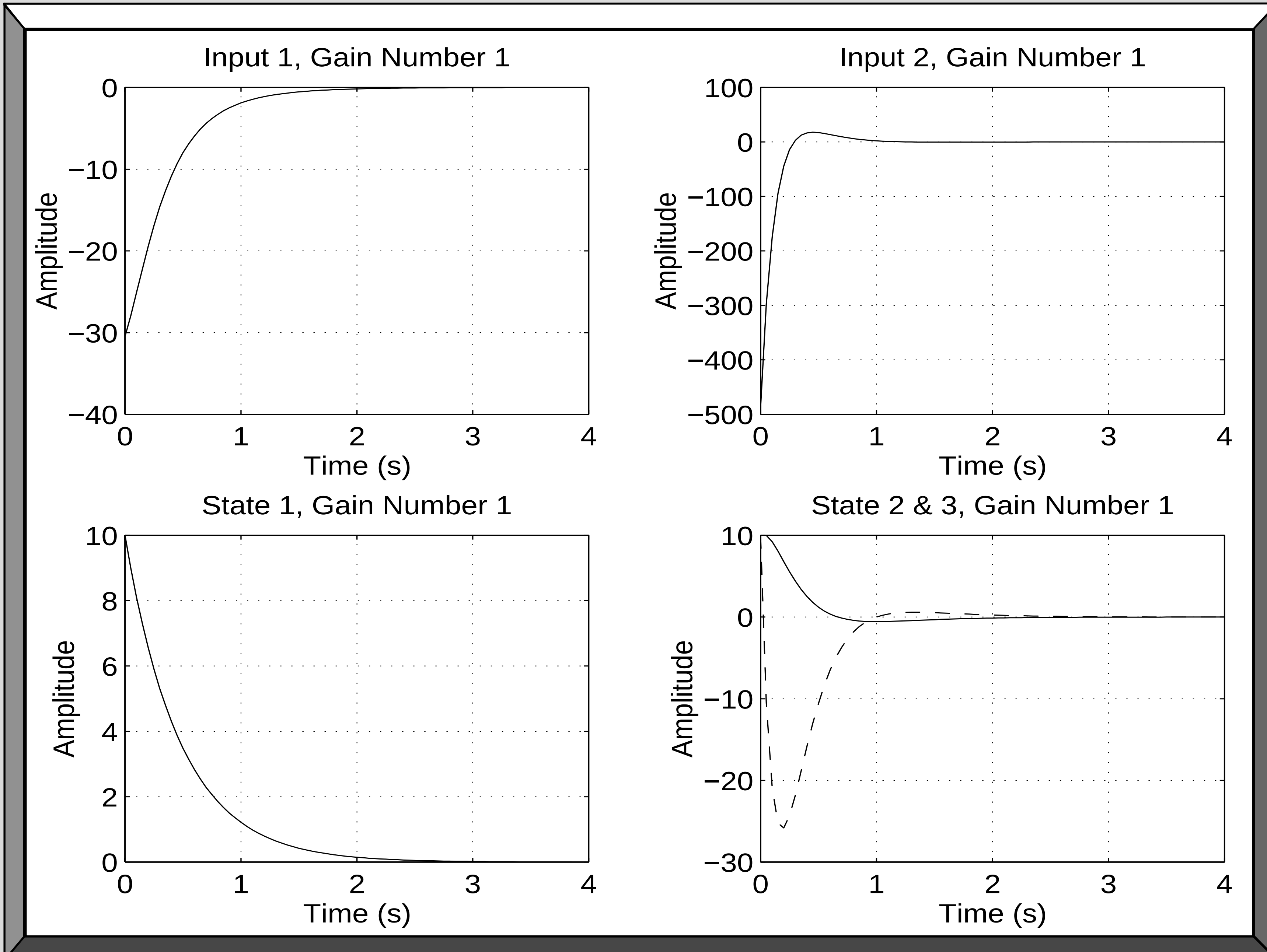
- Discretized system model with T_{samp} = 0.05 seconds:

$$\Phi = \begin{bmatrix} 1.0010 & 0.0501 & 0.0012 \\ 0.0049 & 1.0050 & 0.0489 \\ 0.1954 & 0.2003 & 0.9562 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.0500 & 0.0000 \\ 0.0001 & 0.0012 \\ 0.0049 & 0.0489 \end{bmatrix}, \quad \lambda_{OL} = \begin{bmatrix} 0.9524 \\ 0.9045 \\ 1.1054 \end{bmatrix}$$

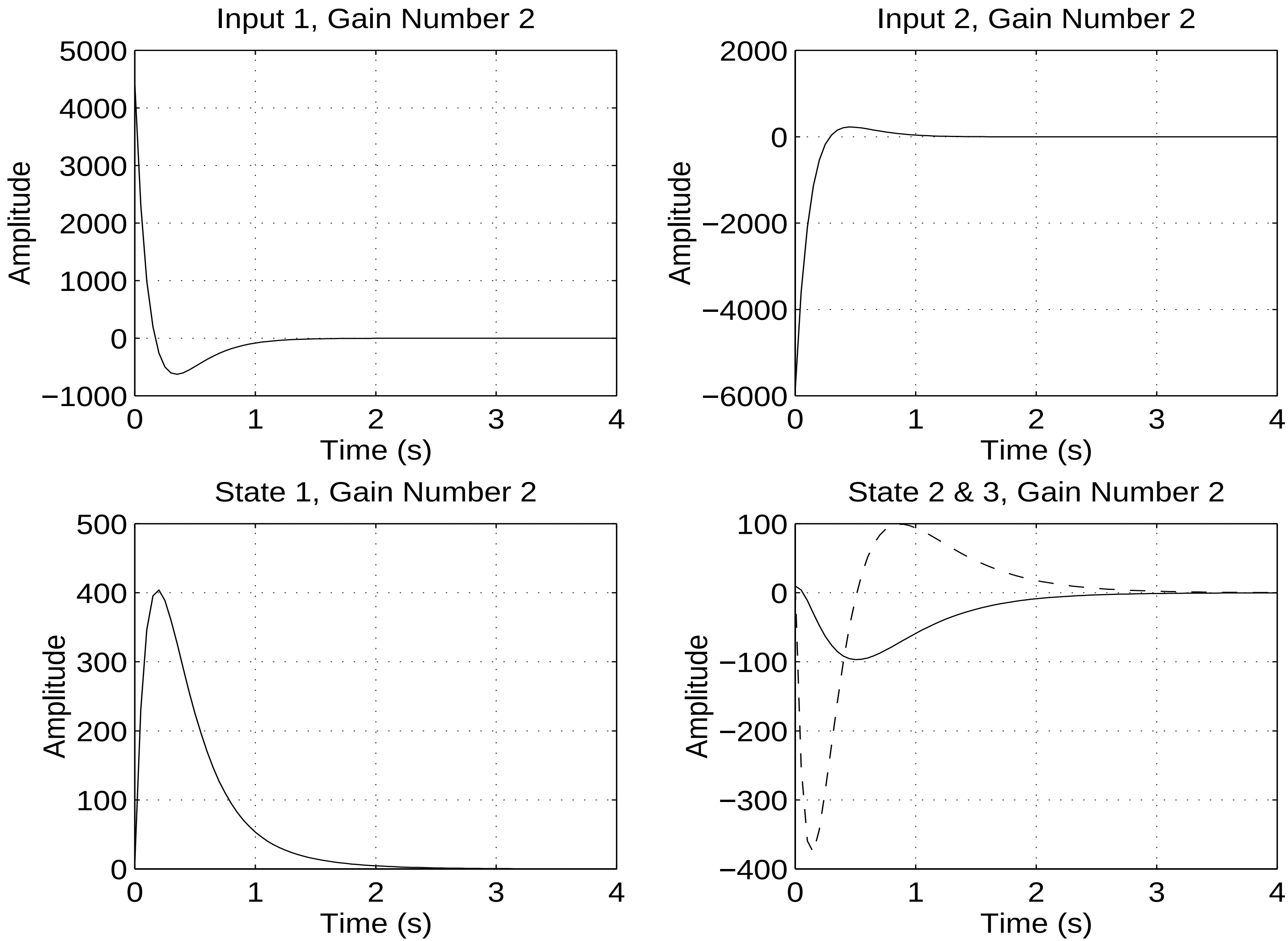
$$x(0) = \begin{bmatrix} 10 & 10 & 10 \end{bmatrix}^T$$

- Desired closed-loop eigenvalues are: λ_{CL} = 0.9, 0.8, 0.7

$$L_1 = \begin{bmatrix} 2.020 & 1.002 & 0.024 \\ 11.525 & 28.568 & 8.726 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -105.74 & -243.56 & -91.52 \\ 142.48 & 323.76 & 121.34 \end{bmatrix}$$



Input 1, Gain Number 1
Amplitude
Time (s)
Input 2, Gain Number 1
Amplitude
Time (s)
State 1, Gain Number 1
Amplitude
Time (s)
State 2 & 3, Gain Number 1
Amplitude
Time (s)


Input 1, Gain Number 2
5000
4000
3000
2000
1000
0
−1000
Amplitude
0
1
2
3
4
Time (s)
Input 2, Gain Number 2
2000
0
−2000
−4000
−6000
Amplitude
0
1
2
3
4
Time (s)
State 1, Gain Number 2
500
400
300
200
100
0
Amplitude
0
1
2
3
4
Time (s)
State 2 & 3, Gain Number 2
100
0
−100
−200
−300
−400
Amplitude
0
1
2
3
4
Time (s)

Motivating Example

- Eigenvalues are identical with each gain matrix.
- Settling times are similar for the two sets of responses.
- Some signal shapes are similar (x_2, x_3, u_2).
- L_2 produces much larger signal amplitudes than L_1 for all variables.
- Quantitative comparison between the 2 responses:

$$J = \sum_{k=0}^{80} \left[x^T(k) Q x(k) + u^T(k) R u(k) \right]$$

$$Q = I_3 \, , \qquad R = I_2$$

$$J_1 = 379{,}450 \, , \qquad J_2 = 84{,}243{,}000$$

Eigenvector Importance

- If the following control law is used:

$$u = -Lx\,, \qquad A_{CL} = A - BL \quad \text{or} \quad \Phi_{CL} = \Phi - \Gamma L$$

- The closed-loop state equations are:

$$x(k) = \Phi_{CL}^{k} x(0)\,, \qquad x(t) = e^{A_{CL} t} x(0)$$

- If the system has real and distinct closed-loop eigenvalues λ_i:

$$\begin{aligned} A_{CL} &= T \Lambda T^{-1} \\ \Phi_{CL} &= T \Lambda T^{-1} \end{aligned}\,, \qquad \Lambda = diag\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix}$$

Eigenvector Importance

- Right eigenvectors for the system are:

$$T = \left[v_1 \vdots v_2 \vdots \cdots \vdots v_n \right], \quad \left[\lambda_i I - A_{CL} \right] v_i = 0 \quad \text{or} \quad \left[\lambda_i I - \Phi_{CL} \right] v_i = 0$$

- Left eigenvectors for the system are:

$$T^{-1} = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix}, \quad w_i^T \left[\lambda_i I - A_{CL} \right] = 0 \quad \text{or} \quad w_i^T \left[\lambda_i I - \Phi_{CL} \right] = 0$$

- The closed-loop system matrices are:

$$\Phi_{CL}^k = T \Lambda^k T^{-1}, \quad e^{A_{CL} t} = T e^{\Lambda t} T^{-1}$$

Eigenvector Importance

- Solutions to the closed-loop state equations are:

$$x(k) = \sum_{i=1}^{n} v_i \lambda_i^k \left[w_i^T x(0) \right], \qquad x(t) = \sum_{i=1}^{n} v_i e^{\lambda_i t} \left[w_i^T x(0) \right]$$

- The eigenvalue λ_i defines the mode of the system.
- Product of left eigenvector and initial conditions $w_i^T x(0)$ determines the amplitude of the mode.
- Right eigenvector v_i determines the shape of the mode.

Finding Eigenvectors

- Each closed-loop right eigenvector satisfies

$$\left[\lambda_i I - A + BL\right] v_i = 0 \quad \text{or} \quad \left[\lambda_i I - \Phi + \Gamma L\right] v_i = 0$$

  - Eigenvector v_i must be in the null space of $\lambda_i I\text{-}A\text{+}BL$ or $\lambda_i I\text{-}\Phi\text{+}\Gamma L$
  - Eigenvectors v_i and v_j must be linearly independent for $i \neq j$.
- But there is a problem: feedback matrix L is unknown!
- Defining $q_i = L v_i$ yields

$$\left[\lambda_i I - A \quad \vdots \quad B\right] \begin{bmatrix} v_i \\ \cdots \\ q_i \end{bmatrix} = 0 \quad \text{or} \quad \left[\lambda_i I - \Phi \quad \vdots \quad \Gamma\right] \begin{bmatrix} v_i \\ \cdots \\ q_i \end{bmatrix} = 0$$

$$v_i = n \times 1, \qquad q_i = m \times 1$$

Finding Eigenvectors

- For each λ_i, the dimension of the null space = m = number of control inputs.
- Singular value decomposition (SVD) can be used to find a basis for the null space.
- For the n closed-loop eigenvalues λ_i

$$[q_1 \;\vdots\; q_2 \;\vdots\; \cdots \;\vdots\; q_n] = L[v_1 \;\vdots\; v_2 \;\vdots\; \cdots \;\vdots\; v_n]$$

$$Q = LT \Rightarrow L = QT^{-1}$$

- That matrix L places eigenvalues and eigenvectors at values specified by λ_i and v_i, respectively.

Finding Eigenvectors

- Singular value decomposition: for each λ_i

$$S_i = \left[\lambda_i I - A \;\vdots\; B\right] = U_i \Sigma_i (V_i)^T \quad \text{or} \quad S_i = \left[\lambda_i I - \Phi \;\vdots\; \Gamma\right] = U_i \Sigma_i (V_i)^T$$

$$S_i = n \times (n+m), \qquad U_i = n \times n$$

$$\Sigma_i = n \times (n+m), \qquad V_i = (n+m) \times (n+m)$$

- Last m columns of V_i are a basis for null space of S_i

$$\begin{bmatrix} v_i \\ \cdots \\ q_i \end{bmatrix} = \sum_{k=1}^{m} \alpha_{i,k} V_{i,n+k}$$

- v_i and v_j must be linearly independent for $i \neq j$

Finding Eigenvectors

Procedure Summarized

- For each desired closed-loop eigenvalue $\lambda_{i:}$
  - Form the matrix S_i ;
  - Form the SVD of S_i to get V_i ;
  - Choose weighting factors $\alpha_{i,k}$ to get v_i and q_i .
- Compute the feedback gain matrix $L = QT^{-1}$

Selecting Eigenvectors

IF T (matrix of eigenvectors) is made to be diagonal by choice of v_i:

THEN

- A_{CL} or Φ_{CL} is diagonal;
- Response of each state variable is controlled by its own eigenvalue;
- Closed-loop system is diagonalized, even though the open-loop system was not diagonal.

Selecting Eigenvectors

- In general, T cannot be made diagonal.

- Alternatives:
  - Choose the v_i so that each eigenvector has as many entries as possible = 0 in off-diagonal positions in *T*, and linear independence is maintained.
  - Choose the v_i so that the eigenvectors are as orthogonal as possible, and linear independence is maintained $\Rightarrow$ MATLAB© "place" function.

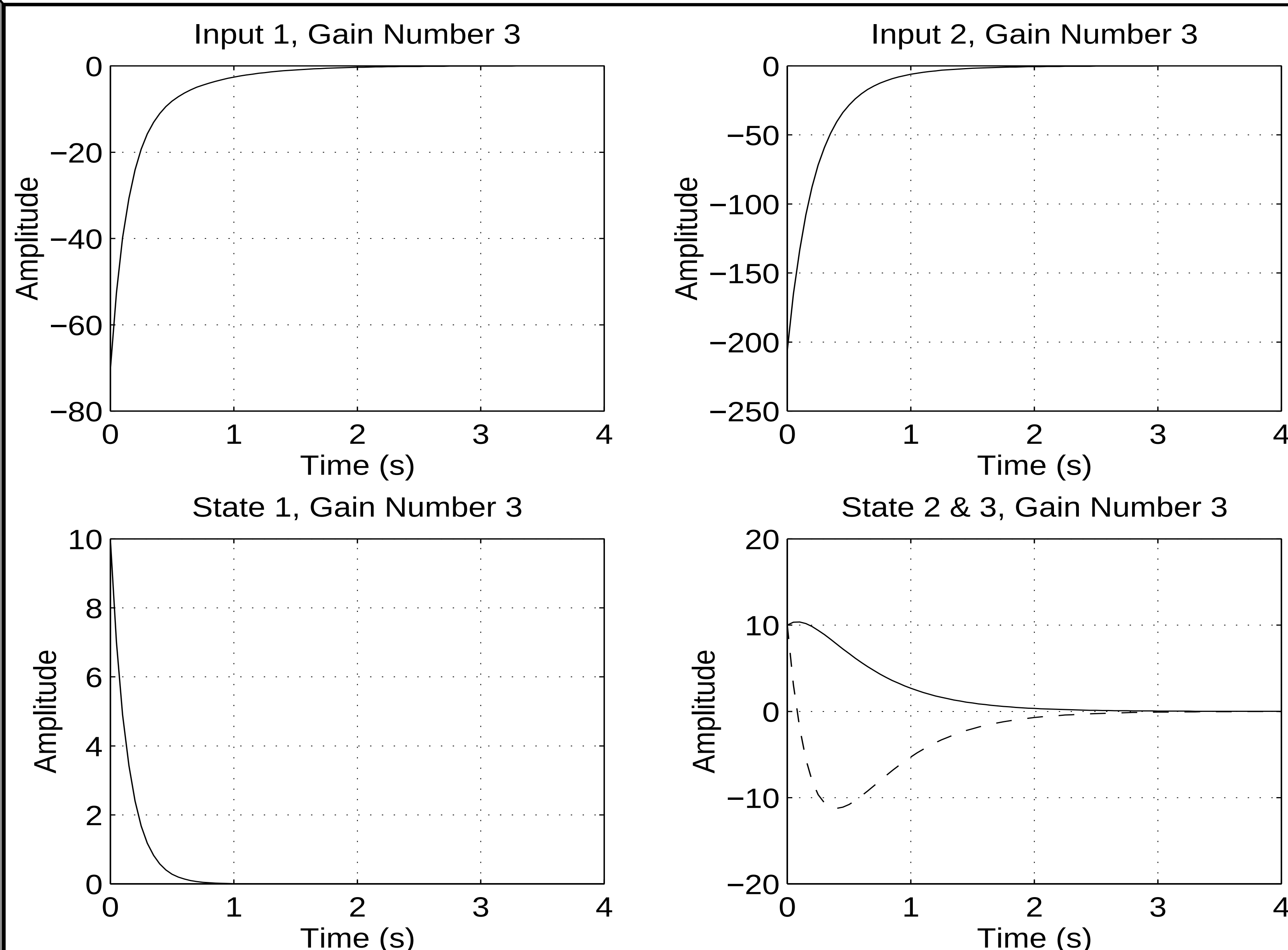
- Neither of these approaches is best in every application.

Example Continued

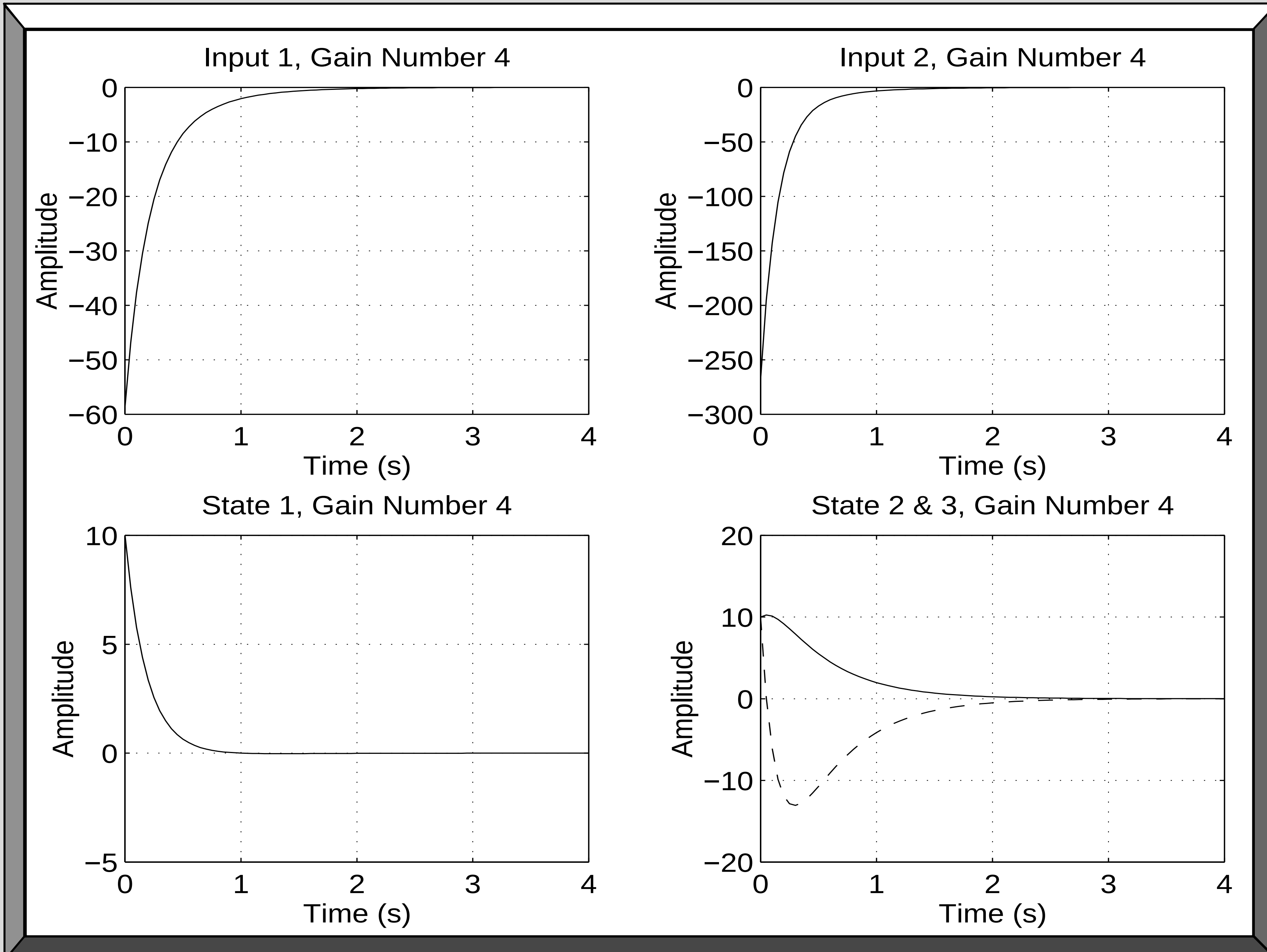
$$V_{1,4-5} = \begin{bmatrix} -0.284 & 0.000 \\ 0.255 & 0.336 \\ -0.536 & -0.709 \\ -0.332 & 0.320 \\ -0.676 & 0.531 \end{bmatrix}, \quad V_{2,4-5} = \begin{bmatrix} -0.216 & 0.000 \\ -0.057 & -0.089 \\ 0.253 & 0.397 \\ -0.920 & -0.080 \\ -0.197 & 0.910 \end{bmatrix}, \quad V_{3,4-5} = \begin{bmatrix} -0.161 & 0.000 \\ -0.015 & -0.030 \\ 0.106 & 0.209 \\ -0.980 & -0.025 \\ -0.048 & 0.977 \end{bmatrix}$$

$$\lambda_1 = 0.9 \qquad \lambda_2 = 0.8 \qquad \lambda_3 = 0.7$$

- L_1: 1st column for 0.9; 2nd column for 0.8 and 0.7.
- L_2: sum of 1st and 2nd columns for 0.9, 0.8, and 0.7.
- L_3: 2nd column for 0.9 and 0.8; $\alpha_{3,k}$ chosen for 0.7 to make the entry in row 3 equal to 0.
- L_4: uses MATLAB© "place" function.



Input 1, Gain Number 3
Input 2, Gain Number 3
State 1, Gain Number 3
State 2 & 3, Gain Number 3
Amplitude
Time (s)
0
−20
−40
−60
−80
−50
−100
−150
−200
−250
10
8
6
4
2
20
−10
−20
1
3


Input 1, Gain Number 4
Amplitude
0
−10
−20
−30
−40
−50
−60
0
1
2
3
4
Time (s)
Input 2, Gain Number 4
Amplitude
0
−50
−100
−150
−200
−250
−300
0
1
2
3
4
Time (s)
State 1, Gain Number 4
Amplitude
10
5
0
−5
0
1
2
3
4
Time (s)
State 2 & 3, Gain Number 4
Amplitude
20
10
0
−10
−20
0
1
2
3
4
Time (s)

Example Continued

$$\Phi_{CL\_1} = \begin{bmatrix} 0.9000 & 0.0000 & 0.0000 \\ -0.0091 & 0.9706 & 0.0384 \\ -0.3781 & -1.2016 & 0.5294 \end{bmatrix}, \quad \begin{matrix} J_1 = 379{,}450 \\ \|L_1\|_2 = 32.058 \end{matrix}$$

$$\Phi_{CL\_2} = \begin{bmatrix} 6.2882 & 12.228 & 4.5771 \\ -0.1555 & 0.6408 & -0.0876 \\ -6.2539 & -14.438 & -4.5290 \end{bmatrix}, \quad \begin{matrix} J_2 = 84{,}243{,}000 \\ \|L_2\|_2 = 467.68 \end{matrix}$$

$$\Phi_{CL\_3} = \begin{bmatrix} 0.7000 & 0.0000 & 0.0000 \\ 0.0002 & 0.9903 & 0.0429 \\ -0.0003 & -0.4008 & 0.7097 \end{bmatrix}, \quad \begin{matrix} J_3 = 138{,}900 \\ \|L_3\|_2 = 13.875 \end{matrix}$$

$$\Phi_{CL\_4} = \begin{bmatrix} 0.7997 & -0.0287 & -0.0116 \\ 0.0001 & 0.9854 & 0.0405 \\ -0.0036 & -0.6006 & 0.6150 \end{bmatrix}, \quad \begin{matrix} J_4 = 169{,}400 \\ \|L_4\|_2 = 18.181 \end{matrix}$$

Example Continued

$$L_1 = \begin{bmatrix} 2.0200 & 1.0220 & 0.0240 \\ 11.525 & 28.568 & 8.7260 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} -105.74 & -243.56 & -91.518 \\ 142.48 & 323.76 & 121.34 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 6.0200 & 1.0220 & 0.0240 \\ 3.3989 & 12.192 & 5.0379 \end{bmatrix}$$

$$L_4 = \begin{bmatrix} 4.0269 & 1.5757 & 0.2561 \\ 3.6665 & 16.219 & 6.9526 \end{bmatrix}$$

Comparison with Optimal LQR

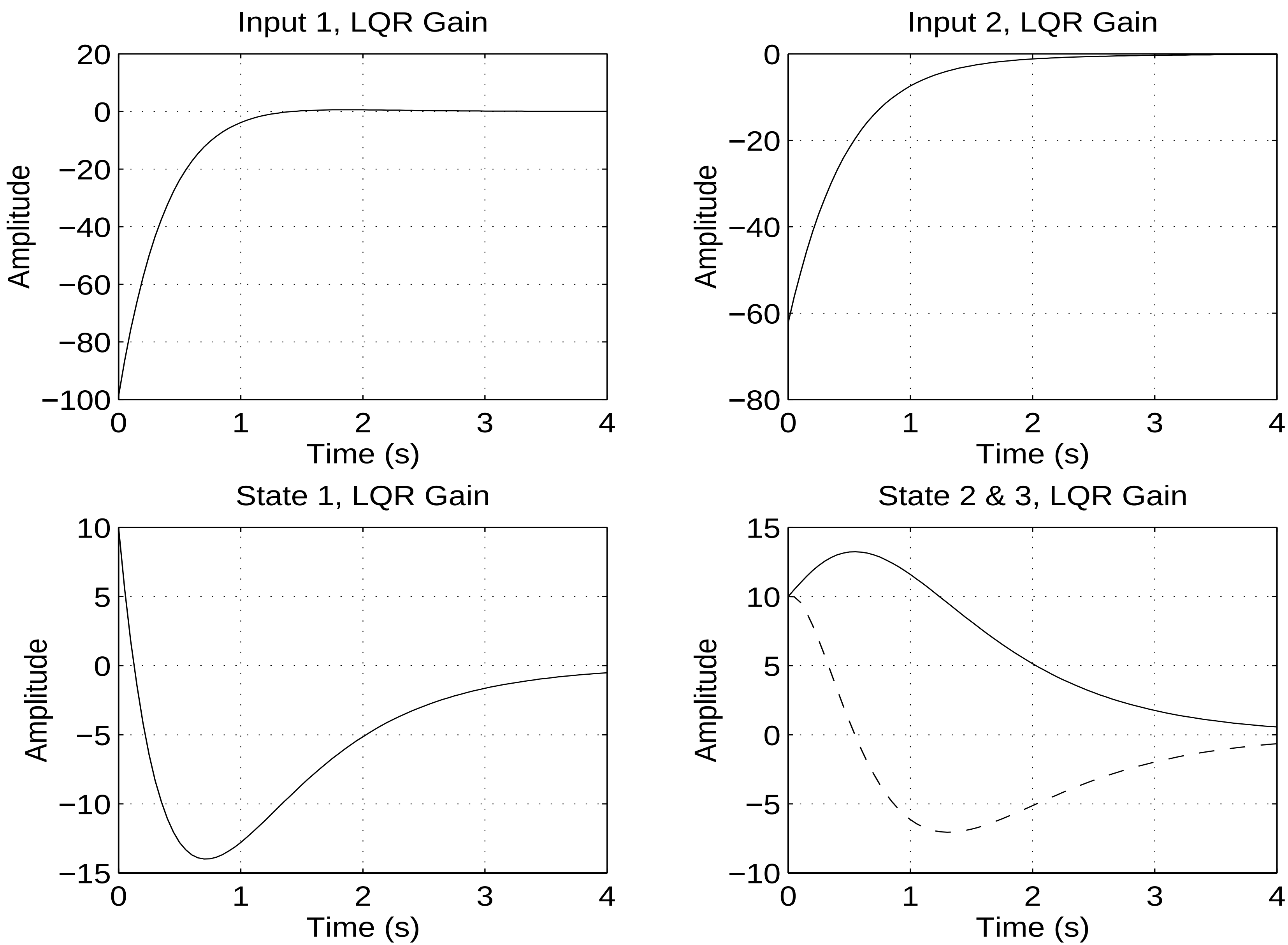
- Weighting Matrices: $Q = I_3$, $R = I_2$

$$\Phi_{CL\_LQR} = \begin{bmatrix} 0.8352 & -0.1938 & -0.0831 \\ 0.0025 & 1.0006 & 0.0472 \\ 0.0945 & 0.0188 & 0.8858 \end{bmatrix}, \qquad J_{LQR} = 71{,}879 \qquad \left\| L_{LQR} \right\|_2 = 7.2451$$

$$L_{LQR} = \begin{bmatrix} 3.3158 & 4.8785 & 1.6856 \\ 1.7306 & 3.2223 & 1.2713 \end{bmatrix}$$

- Closed-loop eigenvalues:

$$\lambda_{CL\_LQR} = \begin{bmatrix} 0.9448 & 0.8884 + j0.0363 & 0.8884 - j0.0363 \end{bmatrix}$$


Input 1, LQR Gain
Input 2, LQR Gain
State 1, LQR Gain
State 2 & 3, LQR Gain
Amplitude
Time (s)
20
0
−20
−40
−60
−80
−100
−80
10
5
−5
−10
−15
15
1
2
3
4

Summary of Results

Gain	J	J/J_{LQR}	$\|L\|_2$	$\|L\|_2 / \|L_{LQR}\|_2$
L_1	379,450	5.28	32.058	4.42
L_2	84,243,000	1,172	467.68	64.6
L_3	138,900	1.93	13.875	1.92
L_4	169,400	2.36	18.181	2.51
L_{LQR}	71,879	1	7.2451	1

- Wide variation in performance is seen, even though the eigenvalues are fixed in location for L_1 - L_4.
- The optimal LQR solution does not guarantee the location of eigenvalues or “satisfactory” performance, only stability and optimality.

Conclusions

- Eigenvectors have a dramatic effect on closed-loop performance.
- For multi-input systems, eigenvectors should be chosen as well as eigenvalues, as far as possible.
- Eigenvectors should be as linearly independent as possible.
- The 2-norm of the feedback gain matrix seems to be a good measure of this independence.