Internal Stability

A. Introduction

Assume that a single-input, single-output closed-loop system consists of the plant $G_p(s)$ in series with a compensator $G_c(s)$ with a unity feedback path, that is, $H(s) = 1$. The normal closed-loop transfer function $T_{CL}(s)$—also known as the complementary sensitivity function $T(s)$—is given by

$$T_{CL}(s) = T(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)} = \frac{N_{CL}(s)}{\Delta_{CL}(s)} \tag{1}$$

where $\Delta_{CL}(s)$ is the closed-loop characteristic equation, the roots of which are the closed-loop poles. The situation described below will not change if feedback compensation rather than series (cascade) compensation is used since $\Delta_{CL}(s)$ is the same for either configuration.

The normal statement on stability is that all roots of $\Delta_{CL}(s)$ must have strictly negative real parts in order for the closed-loop system to be stable. Classical tests such as Bode or Nyquist plots are often used to determine whether or not this requirement is satisfied by investigating the frequency domain plots of the open-loop system $G_c(s)G_p(s)$. Although these tests are generally effective in giving the correct answer, they overlook one potential problem—unstable pole-zero cancellations between the compensator and plant. Unstable poles in $G_p(s)$ that are cancelled by unstable zeros of $G_c(s)$ (or vice-versa) seem to disappear from the system, and they do in fact disappear from the product $G_c(s)G_p(s)$ in terms of the system's frequency response.

Unfortunately, the unstable poles and zeros don't disappear; they are still there and they are still unstable and they are guaranteed to lead to unstable closed-loop poles. Consider a plant $G_p(s)$ with an unstable real pole

$$G_p(s) = \frac{N_p(s)}{D_p(s)} = \frac{N_p(s)}{(s - p_0)\,\overline{D_p}(s)}, \qquad p_0 > 0 \tag{2}$$

and a compensator $G_c(s)$ having an unstable real zero at the same location

$$G_c(s) = \frac{N_c(s)}{D_c(s)} = \frac{(s - z_0)\,\overline{N_c}(s)}{D_c(s)}, \qquad z_0 = p_0 > 0 \tag{3}$$

The closed-loop system corresponding to (1) for this example is

$$T(s) = \frac{\dfrac{(s - z_0)\,\overline{N_c}(s)}{D_c(s)} \cdot \dfrac{N_p(s)}{(s - p_0)\,\overline{D_p}(s)}}{1 + \dfrac{(s - z_0)\,\overline{N_c}(s)}{D_c(s)} \cdot \dfrac{N_p(s)}{(s - p_0)\,\overline{D_p}(s)}} = \frac{(s - z_0)\,\overline{N_c}(s)N_p(s)}{(s - p_0)\,D_c(s)\overline{D_p}(s) + (s - z_0)\,\overline{N_c}(s)N_p(s)} \tag{4}$$

$$= \frac{(s - p_0)\,\overline{N_c}(s)N_p(s)}{(s - p_0)\left[D_c(s)\overline{D_p}(s) + \overline{N_c}(s)N_p(s)\right]} = \frac{N_{CL}(s)}{\Delta_{CL}(s)} = \frac{N_{CL}(s)}{(s - p_0)\,\overline{\Delta_{CL}}(s)} \tag{5}$$

Although it appears that the $(s - p_0)$ in (5) can be cancelled, it must be remembered that $T(s)$ is only the transfer function from one particular input to one particular output, while $\Delta_{CL}(s)$ is the characteristic equation for any input-output pair. For some of those other transfer functions, the cancellation of the unstable pole-zero may not be possible.

In computing the frequency response of $G_c(s)G_p(s)$, the pole/zero cancellation is accomplished, such that Bode and Nyquist methods will not detect the cancellation. Therefore, crucial information can be lost if only the product

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$G_c(s)G_p(s)$ or only the transfer function $T(s)$ is considered. Transfer functions between various inputs and outputs must be considered in order to have a complete picture of stability. The goal is to have every signal in the system remain bounded for any bounded input applied at any point in the system and for any set of initial conditions.

B. The 2-Input, 2-Output System

Since we need to be concerned about intermediate variables in the system, we need a more general block diagram than the one often used where the reference signal is the only input variable and the controlled signal is the only output variable of interest. We can completely analyze the stability problem by introducing the 2-input, 2-output system of Fig. 1 and defining the 2×2 transfer matrix consisting of the four transfer functions between the two external inputs and two of the output signals [1]–[3].

The output signals that we will use are $E(s)$ and $E_1(s)$. The signals $U(s)$ and $C(s)$ could also be used; the end results in terms of stability analysis would be unchanged.

The closed-loop transfer matrix for this system will be designated $H(p, c)$, where p refers to $G_p(s)$ and c refers to $G_c(s)$. The transfer matrix from the inputs $R(s)$ and $R_1(s)$ to the outputs $E(s)$ and $E_1(s)$ is developed by writing the "loop" equation starting at $E(s)$, going around the loop in a counter-clockwise direction, and then solving for $E(s)$, and then repeating the process for $E_1(s)$.

$$\begin{aligned}
E(s) &= R(s) - G_p(s)R_1(s) - G_c(s)G_p(s)E(s) && (6)\\
&\Rightarrow \quad [1 + G_c(s)G_p(s)]\, E(s) = R(s) - G_p(s)R_1(s) && (7)\\
&\Rightarrow \quad E(s) = \frac{R(s) - G_p(s)R_1(s)}{1 + G_c(s)G_p(s)} && (8)\\
E_1(s) &= G_c(s)R(s) + R_1(s) - G_c(s)G_p(s)E_1(s) && (9)\\
&\Rightarrow \quad [1 + G_c(s)G_p(s)]\, E_1(s) = G_c(s)R(s) + R_1(s) && (10)\\
&\Rightarrow \quad E_1(s) = \frac{G_c(s)R(s) + R_1(s)}{1 + G_c(s)G_p(s)} && (11)
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} E(s) \\ E_1(s) \end{bmatrix} &= H(p,c) \begin{bmatrix} R(s) \\ R_1(s) \end{bmatrix} && (12)\\
H(p,c) &= \begin{bmatrix} \dfrac{1}{1 + G_c(s)G_p(s)} & \dfrac{-G_p(s)}{1 + G_c(s)G_p(s)} \\[2ex] \dfrac{G_c(s)}{1 + G_c(s)G_p(s)} & \dfrac{1}{1 + G_c(s)G_p(s)} \end{bmatrix} = \begin{bmatrix} H_{11}(p,c) & H_{12}(p,c) \\[2ex] H_{21}(p,c) & H_{22}(p,c) \end{bmatrix} && (13)
\end{aligned}$$

The concept of *internal stability* is used to catch unstable pole-zero cancellations between the compensator and plant as well as the usual notion of instability being caused by closed-loop poles in the right-half plane (without cancellation). The following section presents the definitions related to internal stability.

C. Stability Definitions and Conditions

Definition 1 (Exponential Stability): A transfer function matrix is exponentially stable if the matrix is proper and has no poles in the closed right-half of the complex s-plane. ♦

For a matrix to be proper—bounded at infinity—each element of the matrix must be proper. This means that the transfer function corresponding to each element of the matrix must have at least as many poles as zeros. From the Smith-McMillan form [1] of a transfer matrix, all poles of the matrix must appear as poles in at least one of the elements of the matrix. Therefore, for a matrix to be exponentially stable, each element of the matrix must be a transfer function that is both stable and proper, that is, each transfer function must be an element of the Hardy space RH_∞ [3]—also known as the ring S [4]. This leads to the definition of internal stability.

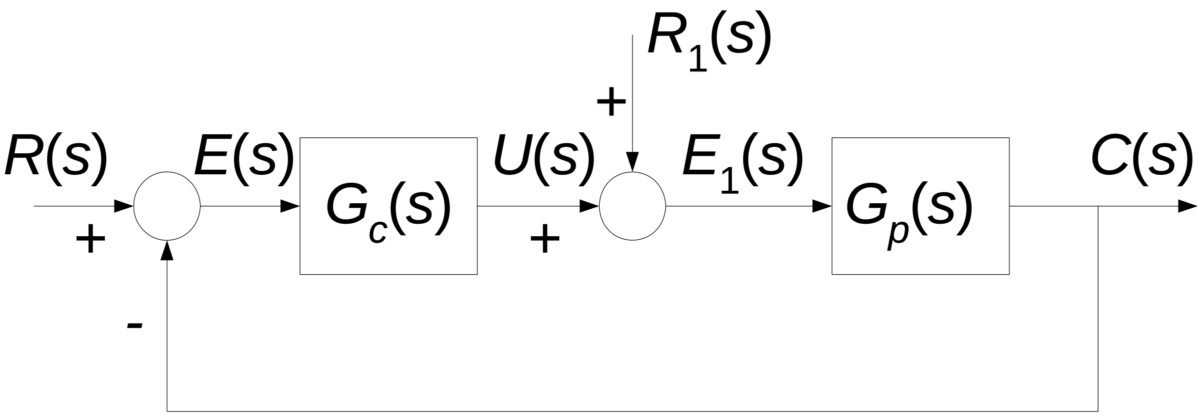

Fig. 1. Block diagram of the 2-input, 2-output system for internal stability analysis.

Definition 2 (Internal Stability): The system represented by the 2-input, 2-output block diagram of Fig. 1 is internally stable if and only if the corresponding transfer matrix $H(p,c)$ given in (13) is exponentially stable. ♦

If there is an unstable pole-zero cancellation between $G_c(s)$ and $G_p(s)$, then either $H_{12}(p,c)$ or $H_{21}(p,c)$ will contain the unstable pole. This indicates that the system is not internally stable since the transfer matrix $H(p,c)$ is not exponentially stable. This is illustrated using the example from the Introduction. Let the plant and compensator be described by (2) and (3), respectively. The four elements of the transfer matrix $H(p,c)$ defined in (13) are

$$H_{11}(p,c) = \frac{E(s)}{R(s)} = \frac{1}{1+G_c(s)G_p(s)} = \frac{1}{1+\frac{(s-z_0)\,\overline{N_c}(s)}{D_c(s)}\cdot\frac{N_p(s)}{(s-p_0)\,\overline{D_p}(s)}} \quad (14)$$

$$= \frac{D_c(s)\overline{D_p}(s)}{D_c(s)\overline{D_p}(s)+\overline{N_c}(s)N_p(s)} \quad (15)$$

$$H_{12}(p,c) = \frac{E(s)}{R_1(s)} = \frac{-G_p(s)}{1+G_c(s)G_p(s)} = \frac{\frac{-N_p(s)}{(s-p_0)\,\overline{D_p}(s)}}{1+\frac{(s-z_0)\,\overline{N_c}(s)}{D_c(s)}\cdot\frac{N_p(s)}{(s-p_0)\,\overline{D_p}(s)}} \quad (16)$$

$$= \frac{-D_c(s)N_p(s)}{(s-p_0)\left[D_c(s)\overline{D_p}(s)+\overline{N_c}(s)N_p(s)\right]} \quad (17)$$

$$H_{21}(p,c) = \frac{E_1(s)}{R(s)} = \frac{G_c(s)}{1+G_c(s)G_p(s)} = \frac{\frac{(s-z_0)\,\overline{N_c}(s)}{D_c(s)}}{1+\frac{(s-z_0)\,\overline{N_c}(s)}{D_c(s)}\cdot\frac{N_p(s)}{(s-p_0)\,\overline{D_p}(s)}} \quad (18)$$

$$= \frac{(s-p_0)\,\overline{N_c}(s)\overline{D_p}(s)}{D_c(s)\overline{D_p}(s)+\overline{N_c}(s)N_p(s)} \quad (19)$$

$$H_{22}(p,c) = \frac{E_1(s)}{R_1(s)} = \frac{1}{1+G_c(s)G_p(s)} = \frac{D_c(s)\overline{D_p}(s)}{D_c(s)\overline{D_p}(s)+\overline{N_c}(s)N_p(s)} = H_{11}(p,c) \quad (20)$$

after possible pole-zero cancellations are performed. The element $H_{12}(p,c)$ is obviously unstable. Therefore, the transfer matrix for the system with this unstable pole-zero cancellation is not exponentially stable, and so the closed-loop system is not internally stable. A bounded input injected at $R_1(s)$ will produce an unbounded signal at $E(s)$ [3].

Note that the unstable pole in $H_{12}(p,c)$ is at the same location as the unstable open-loop pole in $G_p(s)$. If the location of the unstable pole and zero had been reversed between plant and compensator, the element $H_{21}(p,c)$ would have been the unstable element.

The following statements summarize the elements of $H(p,c)$ that must be checked to analyze the internal stability of the closed-loop system.

1) If $G_c(s)$ and $G_p(s)$ are both unstable, then each of the four elements of $H(p,c)$ must be checked for stability. If each of those elements is exponentially stable (stable and proper—an element of $RH_\infty \Leftrightarrow S$), then the closed-loop system is internally stable.
2) If the plant $G_p(s)$ is exponentially stable, then the element of $H(p,c)$ that must be checked is $H_{21}(p,c)$. If that element is exponentially stable, then all elements of the transfer matrix will be exponentially stable, and the closed-loop system is internally stable.
3) If the compensator $G_c(s)$ is exponentially stable, then the element of $H(p,c)$ that must be checked is $H_{12}(p,c)$. If that element is exponentially stable, then all elements of the transfer matrix will be exponentially stable, and the closed-loop system is internally stable.
4) If both the plant $G_p(s)$ and the compensator $G_c(s)$ are exponentially stable, then any element of $H(p,c)$ can be checked. If that element is exponentially stable, then all elements of the transfer matrix will be exponentially stable, and the closed-loop system is internally stable.
5) If any element of $H(p,c)$ is not exponentially stable, then the closed-loop system is not internally stable.

The condition in Statement 3 above, involving the exponential stability of $G_c(s)$, will be proven. The proof of Statement 2 involving $G_p(s)$ is carried out in a similar manner. In regards to Statement 4, if both $G_c(s) \in RH_\infty$ and $G_p(s) \in RH_\infty$, then there can be no unstable pole-zero cancellations. Therefore only the normal check for closed-loop stability needs to be performed.

Proof: [Statement 3] Assume that the compensator $G_c(s)$ is exponentially stable, that is, $G_c(s) \in RH_\infty$. The space $RH_\infty \Leftrightarrow S$ is closed under addition and multiplication [4]. If $H_{12}(s) \notin RH_\infty$, then $H(p,c)$ is not exponentially stable, and the closed-loop system is not internally stable. Therefore, there is no need to check any of the other elements of $H(p,c)$. Assume that $H_{12}(p,c) \in RH_\infty$. Then,

$$1 + G_c(s)H_{12}(p,c) = 1 - \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} \in RH_\infty \quad (21)$$

$$= \frac{1}{1+G_c(s)G_p(s)} = H_{11}(p,c) = H_{22}(p,c) \in RH_\infty \quad (22)$$

which proves that if $H_{12}(p,c) \in RH_\infty$, then $H_{11}(p,c) \in RH_\infty$ and $H_{22}(p,c) \in RH_\infty$.

$$H_{21}(p,c) = \frac{G_c(s)}{1+G_c(s)G_p(s)} = G_c(s) \cdot \frac{1}{1+G_c(s)G_p(s)} = G_c(s) \cdot H_{22}(p,c) \quad (23)$$

so if $G_c(s) \in RH_\infty$ and $H_{22}(p,c) \in RH_\infty$, then $H_{21}(p,c) \in RH_\infty$. From (22), $H_{22}(p,c) \in RH_\infty$ if $H_{12}(p,c) \in RH_\infty$, so with $G_c(s) \in RH_\infty$ being given, the condition that $H_{12}(p,c) \in RH_\infty$ means that $H_{11}(p,c) \in RH_\infty$, $H_{21}(p,c) \in RH_\infty$, and $H_{22}(p,c) \in RH_\infty$. Thus, the stability of $G_c(s)$ and $H_{12}(s)$ guarantees that the transfer matrix $H(p,c)$ is exponentially stable and that the closed-loop system is internally stable. Further, the stability of $G_c(s)$ means that only $H_{12}(s)$ needs to be checked. ■

D. Example

To illustrate the need to have a complete set of information on stability, consider the following plant model $G_p(s)$. A series compensator will be used in a unity feedback configuration.

$$G_p(s) = \frac{8(s+5)}{s(s-2)} \quad (24)$$

The following compensator provides closed-loop stability if the unstable pole-zero cancellation was a valid operation.

$$G_{c_1}(s) = \frac{0.5\,(s-2)}{(s+4)} \tag{25}$$

The resulting complementary sensitivity function is

$$T(s) = \frac{4\,(s+5)\,(s-2)}{(s-2)\,(s+4-j2)\,(s+4+j2)} = \frac{4\,(s+5)}{(s+4-j2)\,(s+4+j2)} \tag{26}$$

Assuming that the cancellation can be done, the resulting second-order system has closed-loop poles at $s = -4 \pm j2$, and so it is stable. If either the Bode or Nyquist plot of $G_{c_1}(s)G_p(s)$ is made—either before or after the cancellation—it would indicate a stable closed-loop system, with an infinite gain margin and a phase margin of $83.6°$. In this case, that would be an incorrect result in terms of internal stability. When the complete $H(p, c_1)$ transfer matrix is formed, we get the following result for the transfer function $H_{12}(p, c_1)$.

$$H_{12}(p, c_1) = \frac{-G_p(s)}{1 + G_{c_1}(s)G_p(s)} = \frac{-\dfrac{8\,(s+5)}{s\,(s-2)}}{1 + \dfrac{0.5\,(s-2)}{(s+4)} \cdot \dfrac{8\,(s+5)}{s\,(s-2)}} \tag{27}$$

$$= \frac{-8\,(s+4)\,(s+5)}{(s-2)\,[s\,(s+4) + 4\,(s+5)]} = \frac{-8\,(s+4)\,(s+5)}{(s-2)\,(s+4-j2)\,(s+4+j2)} \tag{28}$$

This transfer function is clearly unstable, so the closed-loop system is not internally stable. Any signal—even noise—being injected into the system from the input $R_1(s)$ in Fig. 1 will produce an unbounded response at $E(s)$. That unstable response will contain an exponential of the form e^{2t} due to the pole at $s = 2$. The compensator zero at $s = 2$ makes the compensator unresponsive to that unstable exponential; the zero is at the one location guaranteed to make the overall system unstable. The signal injected at $R_1(s)$ will propagate through the unstable plant to reach $E(s)$, but then it will not be processed by the compensator to complete the loop. The control signal $u(t)$ will have no knowledge of the instability, so the unstable signal e^{2t} will remain at the output $e(t)$.

A second compensator will be considered, designed using root locus techniques [5]–[7] as was $G_{c_1}(s)$. This also provides closed-loop stability as indicated by $T(s)$, but there is no unstable pole-zero cancellation involved. This second compensator is

$$G_{c_2}(s) = \frac{0.75\,(s+0.5)}{(s+6)} \tag{29}$$

The complementary sensitivity $T(s)$ with this compensator is

$$T(s) = \frac{6\,(s+5)\,(s+0.5)}{(s+7.4521)\,(s+1.2739-j0.6244)\,(s+1.2739+j0.6244)} \tag{30}$$

which is stable, and the transfer matrix element $H_{12}\,(p, c_2)$ is

$$H_{12}(p, c_2) = \frac{-G_p(s)}{1 + G_{c_2}(s)G_p(s)} = \frac{-\dfrac{8\,(s+5)}{s\,(s-2)}}{1 + \dfrac{0.75\,(s+0.5)}{(s+6)} \cdot \dfrac{8\,(s+5)}{s\,(s-2)}} \tag{31}$$

$$= \frac{-8\,(s+6)\,(s+5)}{s\,(s-2)\,(s+6) + 6\,(s+0.5)\,(s+5)} \tag{32}$$

$$= \frac{-8\,(s+6)\,(s+5)}{(s+7.4521)\,(s+1.2739-j0.6244)\,(s+1.2739+j0.6244)} \tag{33}$$

which is also stable. Since the compensator $G_{c_2}(s)$ is stable and proper, the only element of $H\,(p, c_2)$ that needs to be checked is $H_{12}\,(p, c_2)$. The stability of the transfer function in (33) guarantees that the closed-loop system is internally stable. By contrast, the transfer function in (28) is unstable, so the closed-loop system would not be internally stable if compensator $G_{c_1}(s)$ is used.

Results from the example are shown in Figs. 2 and 3. The root locus diagrams for the systems $G_{c_1}(s)G_p(s)$ and $G_{c_2}(s)G_p(s)$ are shown in Fig. 2. The triangles on the plots indicate the actual closed-loop poles for the respective system. Note that with $G_{c_1}(s)$, there is an open-loop pole, an open-loop zero, and a closed-loop pole at $s = 2$. The closed-loop pole, which moves from an open-loop pole to an open-loop zero as the gain is increased from 0 to ∞, is trapped by the "cancellation" of the pole and zero—they really are not cancelled. All of the closed-loop poles resulting from the use of $G_{c_2}(s)$ are in the left-half plane.

The unit step responses for the various input-output pairs are shown in Fig. 3 for each of the compensators. With compensator $G_{c_1}(s)$, the signal at $e(t)$ resulting from a step input at $r_1(t)$ is unbounded, as expected from (28). All the responses are stable when $G_{c_2}(s)$ is used. The responses from $r_1(t)$ to $e_1(t)$ are not shown since they would be identical to those from $r(t)$ to $e(t)$, as indicated in (20).

E. Stable Pole-Zero Cancellation

The previous sections have defined and illustrated the loss of internal stability when an unstable pole is cancelled by a zero located at the same point in the s-plane. But what about stable pole-zero cancellations; are they valid? The answer is yes—with a caution.

An open-loop pole in the open left-half plane can be cancelled by a corresponding zero without loss of internal stability. However, it should always be remembered that the 'canceled' pole doesn't really disappear, and that it will appear in one or more of the transfer functions in $H(p, c)$. For that reason, it is generally not a good idea to cancel lightly damped complex conjugate open-loop poles—poles that have large imaginary parts relative to their real parts. These poles correspond to small values for the damping ratio ζ. The lightly damped poles produce overshoot and oscillations (ringing) in response to a step input [6], [7].

If the poles to be cancelled are stable and well damped, there will be no problems with either internal stability or time-domain performance if the cancellation is done. This cancellation can actually be an effective design technique in many cases as it allows an open-loop pole to be 'moved' from its original position dictated by the plant to a more favorable location determined by the compensator. However, it should be remembered that this technique should only be applied to stable and well damped poles.

It should also be realized that exact cancellation will never occur in practice. The numerical values of the physical components in the plant are not known precisely, and the values of electronic components (or lines of computer code) used to implement the compensator do not have infinite precision. However, this inexact cancellation should never be used as an excuse to approximate an unstable pole-zero cancellation. Internal stability may not be lost, but designing a compensator to provide closed-loop stability with a near pole-zero cancellation will be very tedious.

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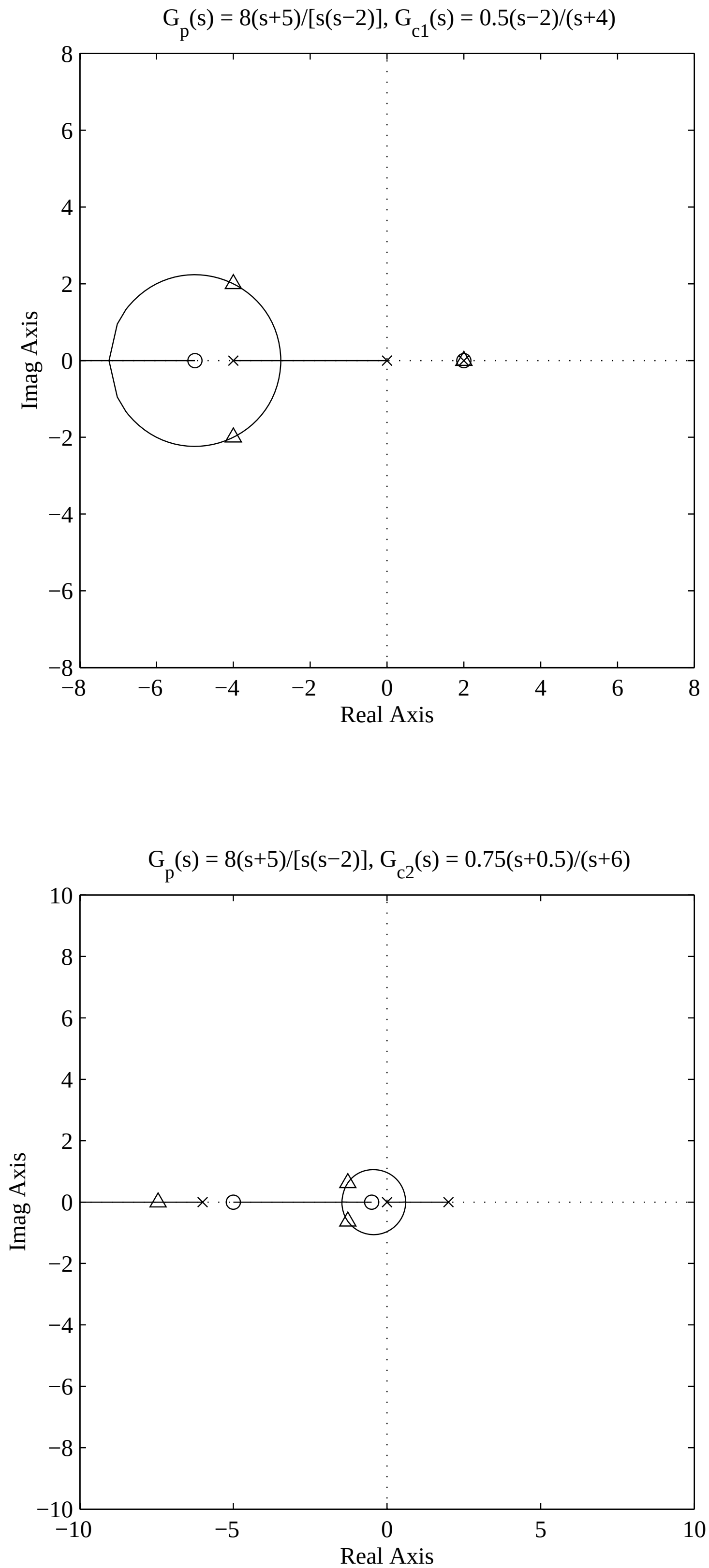

Fig. 2. Compensated root locus diagrams for each of the designs.

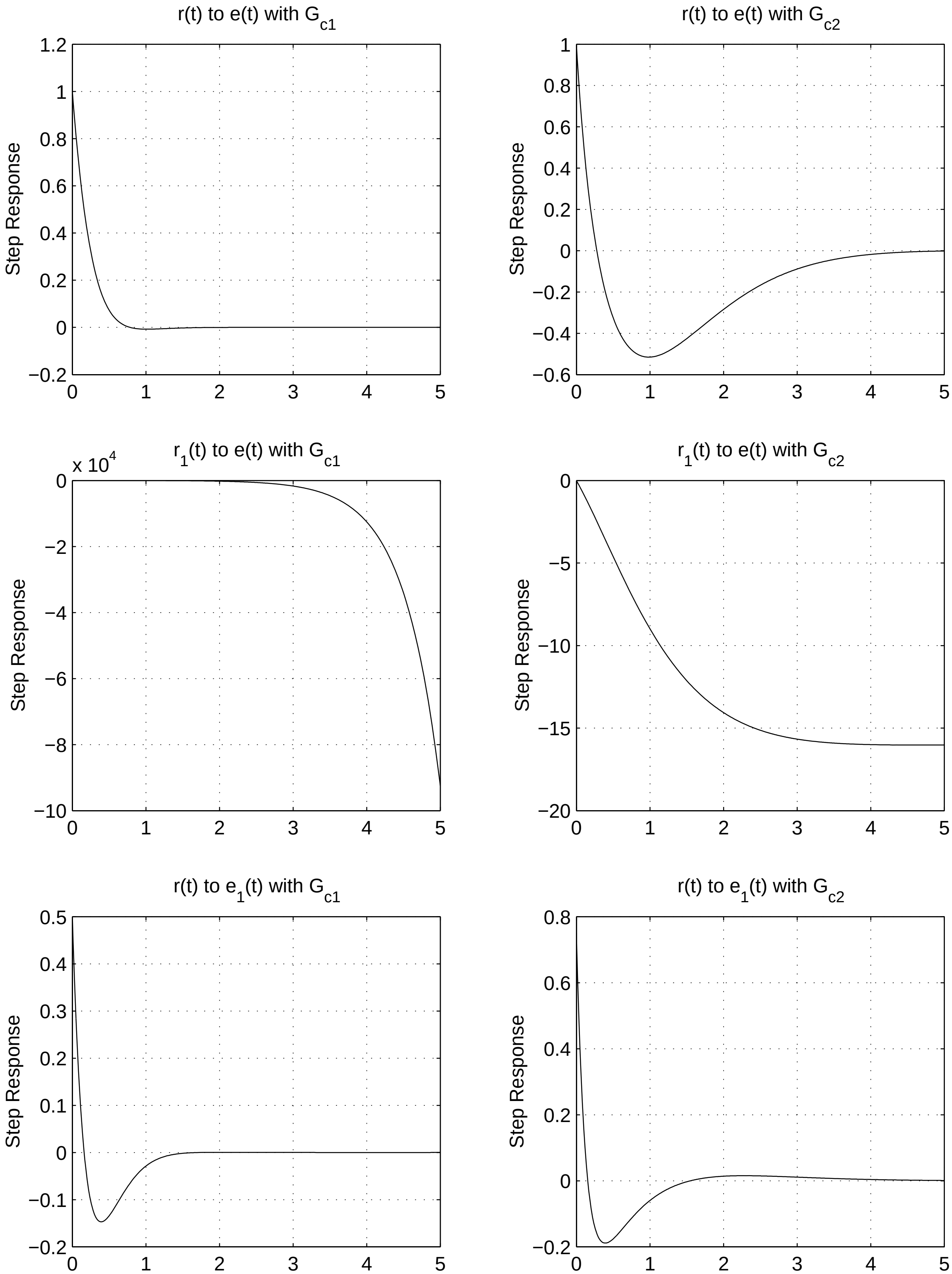

Fig. 3. Compensated step responses for various input-output pairs for the two designs.