Analyzing the Stability Robustness of Interval Polynomials

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I. Introduction

The standard control design problem, at least as it is usually presented in introductory controls courses, is the following: Given a linear model of the system to be controlled in either transfer function or state space format and a set of specifications on performance, design a compensator such that the closed-loop system is stable and all specifications are satisfied. Implicit in this problem statement is that the given mathematical model of the system is an accurate description of the real system, and that this model is not subject to perturbations over time. Both of these assumptions are in general false. Rarely, if ever, do we have a mathematical model for a system that is complete and accurate. Just as the values of resistors and capacitors in electrical circuits have tolerance limits about their nominal values, there is also uncertainty about the values of other types of components in a system, such as spring constants, weights, torque constants, etc. This uncertainty in our knowledge of all the parameter values in a system means that whatever mathematical model we use to represent the system will at best be an approximation to the real system.

Not only is there uncertainty in our knowledge of the system dynamics, but the system will often be subject to perturbations that change those dynamics over time. These perturbations can be due to several factors, including

- changes in parameter values due to aging or stress;
- changes in the operating conditions or environment of the system;
- failure in a system component, such as a sensor or actuator;
- coefficient values that depend explicitly on time.

The changes in parameter values can arise from such things as mechanical wear between parts of a system or from fatigue in a part that is frequently flexed, for example, the suspension system of an automobile. Changes in the linear model representing a system can also occur from a change in operating conditions or the environment in which the system finds itself. This is particularly true for systems that have nonlinear dynamics. For example, the linear model of an airplane changes as its speed and altitude change. Likewise, the model of a ship changes in going from deep water to shallow water. The nonlinear characteristics of actuators will often cause changes in the system model. For example, the rudder of a ship used to control the ship's heading will always have some maximum amount of deflection, such as $\pm 35^\circ$, due to physical limitations. If the controller issues a commanded rudder angle that is less that the limiting value of the rudder, then the rudder will actually achieve that commanded deflection, and the effective gain of the actuator is 1 (in steady-state). However, if the commanded value exceeds the limiting value, the rudder will never achieve its commanded value, so the gain of the actuator is less than 1. Thus, the linear model of the system has changed due to the nonlinear system's response to different signal levels.

Sometimes, the change in the system is due to a failure in a component. A sensor might become damaged during operation and no longer be able to send its signal back to the controller, or its input/output characteristics might change, resulting in the incorrect interpretation of its value. An actuator might also fail in a system. For example the hydraulic actuator for the rudder of a ship might fail, or the rudder itself might become damaged. In either case, the ability to steer the ship would be impaired, and the model representing the ship's heading would change. In electronic circuits, an operational amplifier could be damaged by an over-current situation or excessive temperature. Again, the system model will have changed from its nominal value.

In addition to uncertainty in the model of the system and perturbations to the system itself, there is a another source of differences between the actual system and the mathematical model used to design the compensator. In some situations the actual system model is too complex to be conveniently used for designing the compensator. In these cases, the actual model is approximated by a model of lower order, and there is an intentional difference between the real system and the design model. For example, high frequency bending modes in aircraft and other systems are often neglected during the design of the control system in order to simplify the design process. However, those modes are obviously still part of the real system.

The point being made in each of these examples is that the response of the real system is governed by mathematical relationships that are different from the linear mathematical model that is usually used to design the control system. Whatever the reasons for these differences (lack of knowledge about the system dynamics, perturbations to the dynamics, nonlinearities in the system, intentionally neglected dynamics, etc.), the important point to realize is that the control system must stabilize and produce adequate performance from the real system, not just the model used during the design. This is the task of robust control, namely to design a controller that will satisfy requirements for a family of system models, not just the nominal model. We will use the following definition for a controller in order for it to be classified as a robust controller:

Definition 1: Given a family of system models, consisting of either (1) a finite number of specified models, (2) a nominal model and a description of uncertainty in the model, or (3) a structural model for the system and a description of the uncertainty in the parameter values in the model, then a controller is robust if and only if it internally stabilizes each of the models in the family. ♦

The primary references for the material included in this handout are Barmish [1] and Bhattacharyya, et al. [2]. Other references to robust control include Grimble [3], Skogestad and Postlethwaite [4], and Maciejowski [5]. Two application papers that deal with robust control design and analysis are Beale and Li [6] and Montanaro and Beale [7].

II. Analysis of Interval Polynomials

A. Overview

As with all questions concerning stability of control systems, the polynomial that we are interested in is the closed-loop characteristic equation $\Delta_{CL}(s)$, the denominator of the closed-loop transfer function $T(s)$. Given the forward-path and feedback-path transfer functions $G(s)$ and $H(s)$, this characteristic equation is $\Delta_{CL}(s) = 1 + G(s)H(s) = 0$. With $G(s)H(s)$ expressed as a ratio of polynomials $N(s)/D(s)$, the characteristic equation is $\Delta_{CL}(s) = D(s) + N(s) = 0$. Based on the discussion in the previous section, perturbations to the system or uncertainties in system parameter values result in uncertainties in the coefficients of $\Delta_{CL}(s)$. We want to determine whether or not the closed-loop system remains BIBO stable for all allowed values of the polynomial's coefficients.

We will discuss only the simplest case here, namely the case where the coefficients of $\Delta_{CL}(s)$ are allowed to vary independently of one another. Thus, if the characteristic equation is

$$\Delta_{CL}(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \tag{1}$$

we will assume that each of the coefficients can vary between a lower bound and an upper bound and that there is no dependency between the variations in the coefficients. A polynomial that has independent variations in its coefficients is known as an *interval polynomial*. Although this assumption of independent variation is not realistic in most cases, it is the starting point for studying stability robustness where there are dependencies among the coefficients' variations. The assumption of independent variation also provides an over-bounding of any more complicated types of uncertainty, so it provides a worst-case analysis of stability robustness.

For the sake of simplicity, we will also assume that the leading coefficient a_n is constant and the coefficients have been normalized so that $a_n = 1$. The polynomial coefficients can then be expressed as

$$a_i \in \left[a_i^{\min},\ a_i^{\max}\right], \quad i = 0,\, 1,\, \cdots,\, n-1 \tag{2}$$

so the characteristic equation that we will study is

$$\Delta_{CL}(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \tag{3}$$

The question to be answered is: are all the zeros of the polynomial $\Delta_{CL}(s)$ (closed-loop poles of the system) strictly in the left-half of the complex s-plane for all values of the a_i coefficients within the bounds shown in (2)? If the answer is yes, then the family of systems described by (3) is robustly stable. For example, consider a third-degree polynomial $\Delta_{CL}(s)$ in the form of (3) with the following bounds on the uncertainty:

$$a_0 \in [38,\ 58]\,, \qquad a_1 \in [25,\ 39]\,, \qquad a_2 \in [8,\ 12]\,, \qquad a_3 = 1 \tag{4}$$

The "box of uncertainty" for the coefficients $\{a_0,\, a_1,\, a_2\}$ is shown in Fig. 1. With independent variations in each of the three coefficients, any point inside the box or on its boundary represents a valid set of coefficients for this characteristic equation. Thus, there is a triply-infinite set of characteristic equations that must be checked for stability. This example will be continued in later paragraphs.

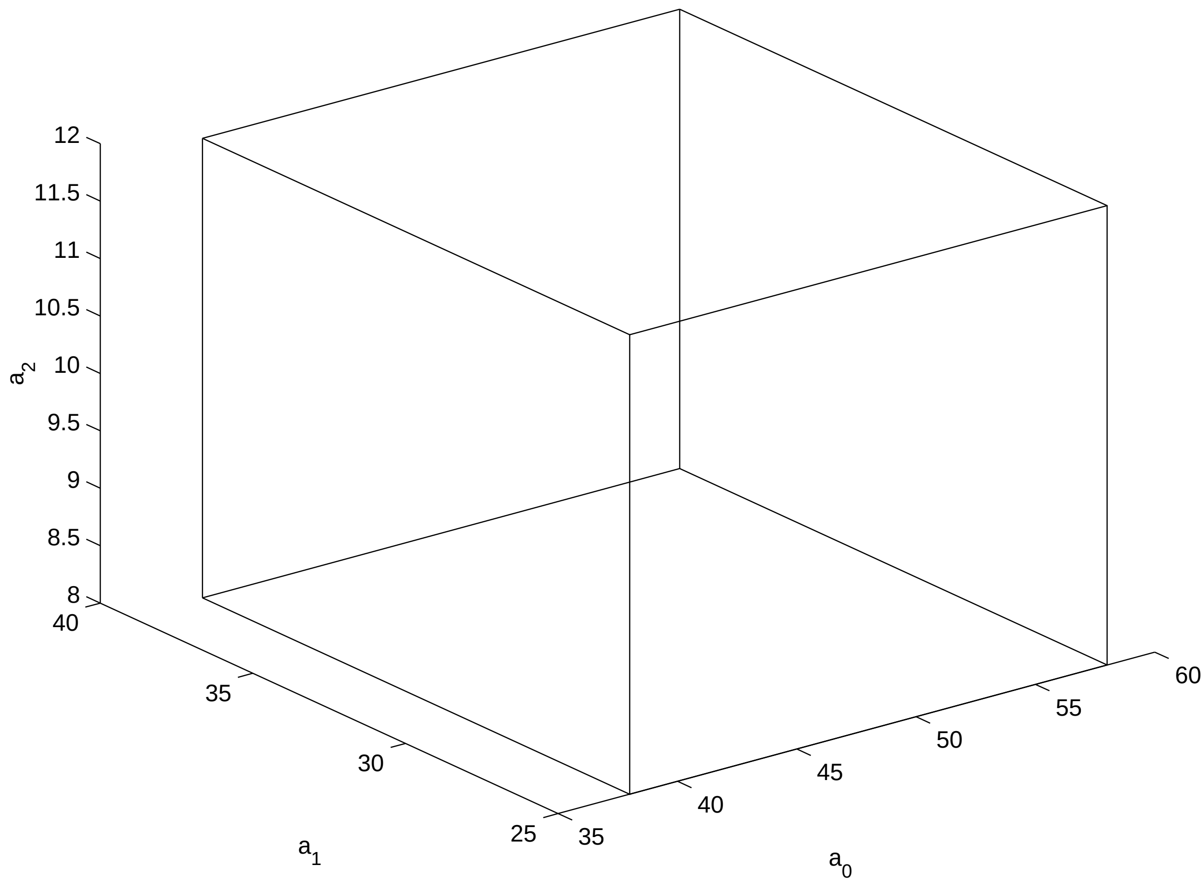

Fig. 1. Box of uncertainty for 3rd-order example characteristic equation.

B. Boundary Crossing, Zero Exclusion, and Value Sets

Assume that there is a nth-degree polynomial with $a_n = 1$ and independent variations in the coefficients $A = \{a_{n-1}, \cdots, a_1, a_0\}$. Since the most significant coefficient a_n is never equal to zero in this case, all the polynomials in the family are of degree n. Now assume that for at least one set of coefficients, $A^a = \left\{a^a_{n-1}, \cdots, a^a_1, a^a_0\right\}$, the closed-loop system is stable. If it is true that there is another set of coefficients $A^b = \left\{a^b_{n-1}, \cdots, a^b_1, a^b_0\right\}$ such that there is at least one closed-loop pole in the right half plane, then it is also true that there is a set of coefficients $A^c = \left\{a^c_{n-1}, \cdots, a^c_1, a^c_0\right\}$ such that there are no closed-loop poles in the right-half plane and that there is at least one pole on the $j\omega$ axis. What this says is that in order for a system to go from having all left-half plane roots to having at least one root in the right-half plane, there must be the condition where there is one or more roots on the boundary of stability but none in the right-half plane. This is known as the *boundary crossing* phenomenon. Thus, to go from stable to unstable, you have to cross the boundary of stability ($j\omega$ axis), assuming that all polynomials in the family are of degree n (and that the a_i coefficients vary continuously between their upper and lower bounds).

Therefore, if it was known that there was at least one stable polynomial in the family of $\Delta_{CL}(s)$, the loss of robust stability could be detected by evaluating each polynomial in the family along the $j\omega$ axis and determining if any of the polynomials evaluated to 0 at any frequency $\omega = \omega_1$. If $\Delta_{CL}(j\omega_1) = 0$ for some set of coefficients A^c, then that polynomial has one or more roots on the $j\omega$ axis, and it has been demonstrated that the system is not robustly stable. The test for robust stability then becomes the evaluation (at least conceptually) along the boundary of stability of every polynomial in the family and determining if 0 is included in any of results. If 0 is not present in any of the evaluations (0 is excluded from the results), then the family is robustly stable. This is known as the *zero exclusion* principle, and it serves as the basis for the experimental determination of robust stability.

Assume that for our family of nth-degree polynomials we could evaluate each one of them along the boundary of stability. At each point on the boundary, each of the polynomials would evaluate to a complex number. For example at frequency $\omega = \omega_1$, the polynomial with coefficients A^a would evaluate to $\Delta^a_{CL}(j\omega_1) = x^a_1 + jy^a_1$. As frequency varies and as the polynomial coefficients vary, the complex number changes. This set of complex numbers can be plotted in a 2-dimensional plane (a complex plane, but not the s-plane). The *value set* is defined to be all the complex numbers generated at a particular frequency $\omega = \omega_1$ by the family of polynomials $\Delta_{CL}(j\omega_1)$ as the coefficients of the polynomial vary over their allowed ranges. The value set can be represented by a polygon in the complex plane. As frequency varies, the value set moves through the plane. Thus, the value set has mapped the stability analysis of an nth-degree polynomial family into the complex plane, which is always two-dimensional. Combining the concepts of boundary crossing, zero exclusion, and value sets, robust stability analysis of a polynomial family becomes determining whether or not 0 is excluded from the family of value sets as the polynomials are evaluated along the stability boundary. If the family of n^{th}-degree polynomials is robustly stable, the value sets will move in a counter-clockwise direction through n quadrants of the complex plane without passing through or touching the origin of the plane.

C. Kharitonov Polynomials

Fortunately in the case of interval polynomials, not every polynomial in the family needs to be individually tested for zero exclusion. Since each coefficient that varies generates an infinite number of polynomials, this is good news. In 1978 the Russian mathematician Kharitonov proved that, for interval polynomials, there are at most 4 polynomials that have to be tested, regardless of the degree n of the polynomial family. Thus, there is an immense savings in computation when the polynomial family has the interval uncertainty structure.

The coefficients of the four Kharitonov polynomials depend only on the upper and lower bounds of the corresponding coefficients in the polynomial family. Therefore, these polynomials are known in advance. The following result applies to interval polynomial families. Note that these results are necessary and sufficient.

Theorem 2: An nth-degree interval polynomial family described by (2) and (3) is robustly stable if and only if each of the four Kharitonov polynomials is stable, that is, all the roots of those polynomials have strictly negative real parts. ♦

Therefore, robust stability of an interval polynomial family can be determined by forming the four Kharitonov polynomials, factoring them, and examining the roots. If all the roots are in the left-half plane, then the family is robustly stable. If any of those polynomials has roots on the $j\omega$ axis or in the right-half plane, the family is not robustly stable. The Kharitonov polynomials are given by

$$
\begin{aligned}
K_1(s) &= a_0^{\min} + a_1^{\min} s + a_2^{\max} s^2 + a_3^{\max} s^3 + a_4^{\min} s^4 + a_5^{\min} s^5 + a_6^{\max} s^6 + a_7^{\max} s^7 + \cdots \qquad (5)\\
K_2(s) &= a_0^{\max} + a_1^{\max} s + a_2^{\min} s^2 + a_3^{\min} s^3 + a_4^{\max} s^4 + a_5^{\max} s^5 + a_6^{\min} s^6 + a_7^{\min} s^7 + \cdots \\
K_3(s) &= a_0^{\max} + a_1^{\min} s + a_2^{\min} s^2 + a_3^{\max} s^3 + a_4^{\max} s^4 + a_5^{\min} s^5 + a_6^{\min} s^6 + a_7^{\max} s^7 + \cdots \\
K_4(s) &= a_0^{\min} + a_1^{\max} s + a_2^{\max} s^2 + a_3^{\min} s^3 + a_4^{\min} s^4 + a_5^{\max} s^5 + a_6^{\max} s^6 + a_7^{\min} s^7 + \cdots
\end{aligned}
$$

When these Kharitonov polynomials are evaluated at a point on the $j\omega$ axis, they form the four corners of a rectangle whose edges are parallel with the real and imaginary axes. With the polynomial definitions in (5), $K_1(j\omega)$ is the lower left corner, $K_2(j\omega)$ is the upper right corner, $K_3(j\omega)$ is the lower right corner, and $K_4(j\omega)$ is the upper left corner. This rectangle is the value set for the polynomial family at that particular frequency. For the 3^{rd}-order example started in the previous section, the Kharitonov polynomials are

$$\begin{aligned} K_1(s) &= 38 + 25s + 12s^2 + s^3, \qquad K_2(s) = 58 + 39s + 8s^2 + s^3 \\ K_3(s) &= 58 + 25s + 8s^2 + s^3, \qquad K_4(s) = 38 + 39s + 12s^2 + s^3 \end{aligned} \tag{6}$$

and the value sets for $\omega = 1$ r/s and $\omega = 2$ r/s are shown in Fig. 2. Factoring the polynomials shows that all the roots are in the left half plane, so this family of polynomials is robustly stable. If the value sets were plotted for all frequencies, 0 would be excluded from them, providing a graphical conclusion of robust stability.

D. Example

The 3^{rd}-order example will be examined in more detail in this section. Assume that the system to be controlled (plant) and its compensator are given by the transfer functions

$$G_c(s) = \frac{4\,(s+3)}{(s+8)}, \qquad G_p(s) = \frac{4}{s\,(s+2)} \tag{7}$$

The closed-loop characteristic equation for this system is

$$\Delta_{CL}(s) = s^3 + 10s^2 + 32s + 48 \tag{8}$$

which is a stable polynomial having roots at $s = -2 \pm j2, -6$. Now assume that the parameter values in the plant and/or compensator are subject to perturbations, so that the coefficients in the characteristic equation change. We want to know if stability is maintained as the parameters change values.

We will look at four cases, with the coefficient intervals being

$$\begin{aligned} &\text{Case \#1}: \quad a_0 \in [43.2,\ 52.8]\,, \quad a_1 \in [28.8,\ 35.2]\,, \quad a_2 \in [9,\ 11] \\ &\text{Case \#2}: \quad a_0 \in [38.4,\ 57.6]\,, \quad a_1 \in [25.6,\ 38.4]\,, \quad a_2 \in [8,\ 12] \\ &\text{Case \#3}: \quad a_0 \in [24,\ 72]\,, \quad a_1 \in [16,\ 48]\,, \quad a_2 \in [5,\ 15] \\ &\text{Case \#4}: \quad a_0 \in [19.2,\ 76.8]\,, \quad a_1 \in [12.8,\ 51.2]\,, \quad a_2 \in [4,\ 16] \end{aligned} \tag{9}$$

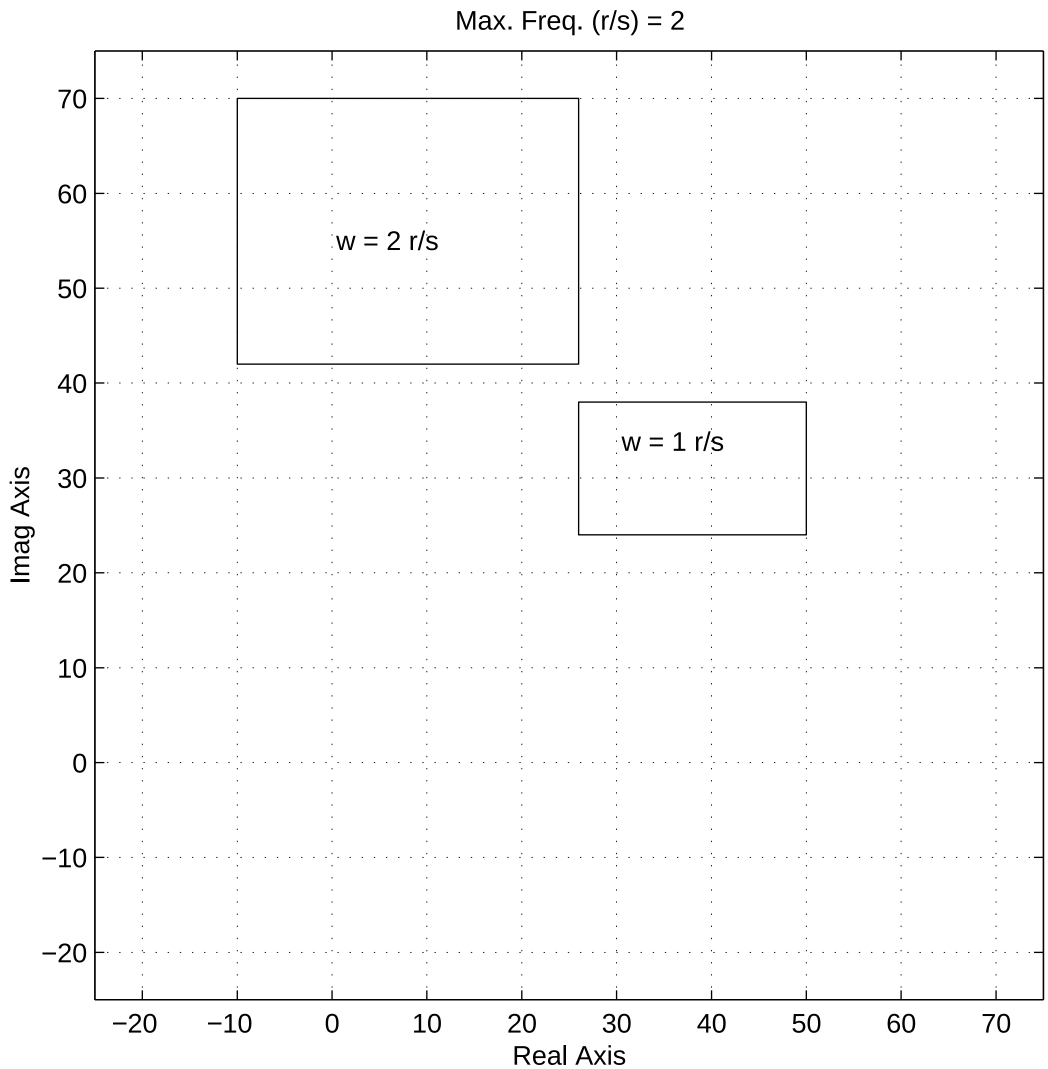

Fig. 2. Two Kharitonov rectangles for the 3rd-order example.

These cases correspond to perturbations of the nominal polynomial coefficients by $\pm 10\%$, $\pm 20\%$, $\pm 50\%$, and $\pm 60\%$, respectively. The most significant coefficient $a_3 = 1$ in each case. Case #2 is approximately the same as shown in the earlier discussion of the example. The Kharitonov polynomials for the various cases could be formed and factored to determine whether or not the family was robustly stable for those perturbations. We will investigate the robustness by plotting the value sets.

Since our polynomial families are 3^{rd}-order, the value sets will move through 3 quadrants in a counter-clockwise direction and exclude the origin if the family is robustly stable. The value sets for the four cases are shown in Fig. 3. The ordering of the cases in the subplots is upper left, upper right, lower left, and lower right.

The value sets for each of the first three sets move from the first quadrant to the second and then to the third quadrant without passing through the origin. Therefore, each of these families is robustly stable. The value sets for the fourth case, corresponding to $\pm 60\%$ variations in the coefficients, pass through the origin, so that family is not robustly stable. Although the rectangles are in the third quadrant for $\omega = 7.5$ r/s, some parts of the rectangle took a "shortcut" to get there, going through the fourth quadrant rather than the second quadrant. Specifically, the lower right corner of the rectangle, $K_3(j\omega)$ and polynomials close to it, took the shortcut. The example shows that for this particular plant and compensator, stability can be maintained for independent variations in the characteristic equation coefficients up to approximately $\pm 50\%$ of the nominal coefficients.

III. Conclusions

Since no mathematical model of a system will be a perfect representation of the actual system, a study of stability robustness is an important task in control system design. The controller must stabilize and provide good performance for the real system, not just its mathematical model. No matter what technique has been used to design the controller, an analysis of robustness should be made when the design is complete. This can be done to determine the minimum size of perturbation that will destabilize the system or to see if stability is maintained for a particular set of assumed perturbations.

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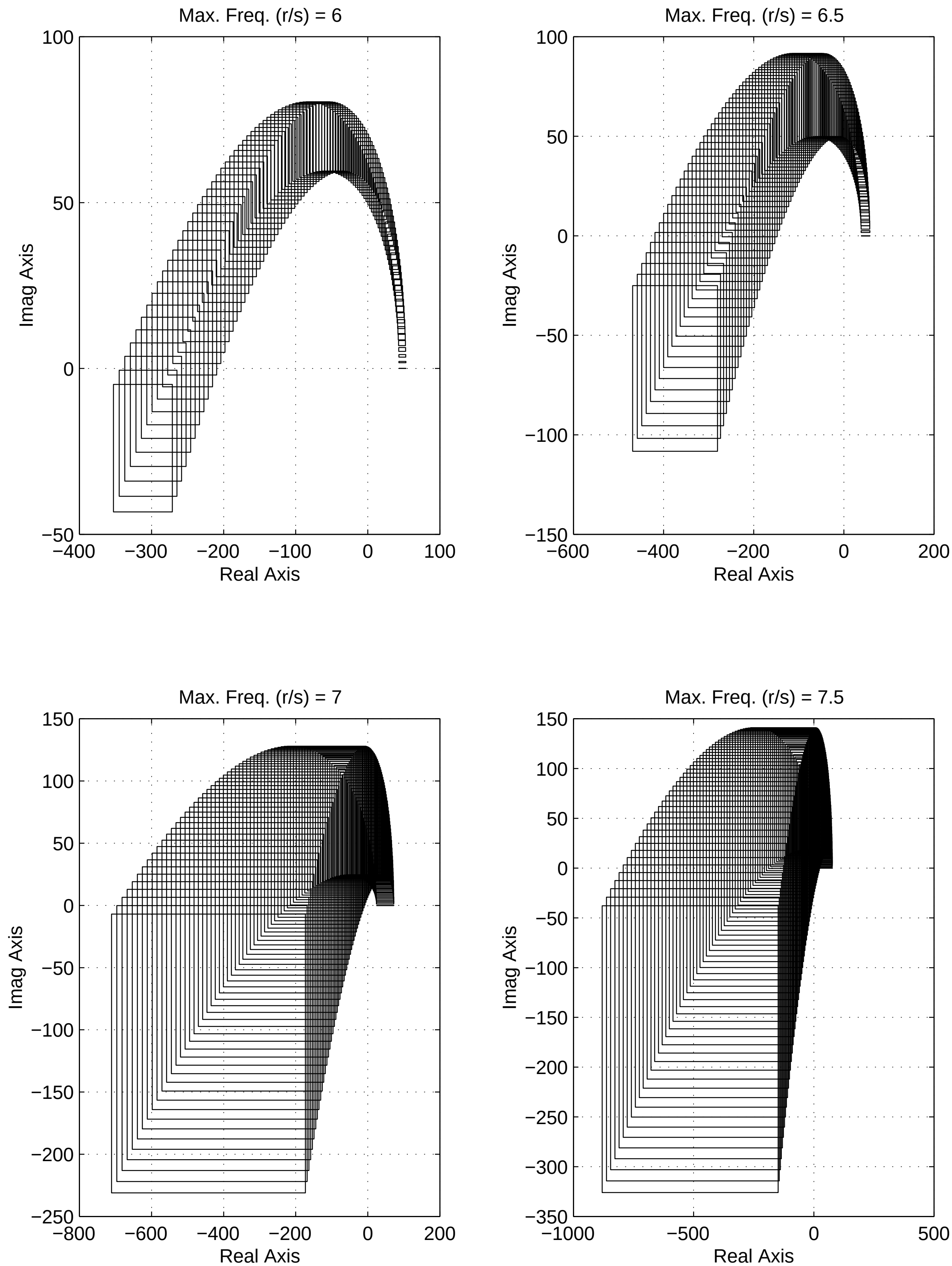

Fig. 3. Values sets for the 4 cases in the example.