Step Response Due To Non-Minimum-Phase Zeros

A. System Model

Assume that the system is modeled as a Single-Input, Single-Output (SISO) configuration with unity feedback and the following forward loop transfer function

$$G(s) = \frac{N(s)}{D(s)} = \frac{(s-z_0)\,\bar{N}(s)}{s\bar{D}(s)} \tag{1}$$

$$z_0 > 0, \qquad \bar{D}(z_0) \neq 0 \tag{2}$$

$$\bar{N}(0) \neq 0 \tag{3}$$

The expressions in (1) and (2) indicate that one open-loop zero of the system is in the right-half of the complex s-plane (RHP) and that there is no pole-zero cancellation involving that term. The expressions in (1) and (3) indicate that the system is Type 1, having one open-loop pole at the origin, and there is no pole-zero cancellation of that term. The assumption that the system is Type 1 (or higher) is not a strict requirement; it is included here merely as a matter of convenience in the proof that follows.

With the open-loop system $G(s)$ defined in (1)–(3), the closed-loop transfer function is

$$T_{CL}(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{(s-z_0)\,\bar{N}(s)}{s\bar{D}(s)}}{1+\dfrac{(s-z_0)\,\bar{N}(s)}{s\bar{D}(s)}} = \frac{(s-z_0)\,\bar{N}(s)}{s\bar{D}(s)+(s-z_0)\,\bar{N}(s)} = \frac{(s-z_0)\,\bar{N}(s)}{\Delta_{CL}(s)} \tag{4}$$

It will be assumed that the closed-loop system is internally stable. Therefore, all roots of the closed-loop characteristic equation $\Delta_{CL}(s)$ lie strictly in the left-half of the complex s-plane (LHP), and there are no unstable pole-zero cancellations in $G(s)$—there are no unstable hidden modes in the system.

The reference input to the system will be the unit step function, so the transform of the reference input is

$$R(s) = \frac{1}{s} \tag{5}$$

The transform of the system output is the product of the transform of the reference input signal and the closed-loop transfer function. Therefore, the transform of the output signal is

$$Y(s) = T_{CL}(s)R(s) = \frac{(s-z_0)\,\bar{N}(s)}{s\Delta_{CL}(s)} \tag{6}$$

B. Region of Convergence

The single-sided (unilateral) Laplace transform for a signal $x(t)$ is defined to be

$$\pounds\left[x(t)\right] = X(s) = \int_0^{\infty} x(t)e^{-st}dt \tag{7}$$

assuming that the integral can be evaluated at the upper and lower limits to yield well-defined and bounded values. Associated with this transform $X(s)$, and equivalent to the statement that the transform exists, is the signal's Region of Convergence (ROC). If the transform exists, the ROC exists, and vice versa. The ROC is that open half of the s-plane that lies to the right of all singularities (poles) of $X(s)$. At any point $s = s_0$ that lies inside the ROC, the following relationship is true.

$$X\left(s_0\right) = \int_0^{\infty} x(t)e^{-s_0 t}dt \tag{8}$$

Therefore, at any point $s = s_0$ in the Region of Convergence, the form of the defining equation of the Laplace transform is still applicable, with the complex variable s replaced by the complex number s_0. The result of evaluating (8) is a numerical value that is a real number if s_0 is real and a complex number if s_0 is complex.

In the problem being considered here, the closed-loop system is assumed be to internally stable. All roots of $\Delta_{CL}(s)$ lie in the open left-half plane. Thus, the Region of Convergence for the transform $Y(s)$ in (6) is the open right-half of the s-plane. Therefore, the open-loop zero z_0 lies in the ROC for $Y(s)$, and (8) can be evaluated for $Y(s)$ and $y(t)$ with $s_0 = z_0$.

C. Form of the Step Response

Applying (8) to $Y(s)$ and $y(t)$ with $s_0 = z_0$ yields the following result

$$\int_0^\infty y(t)e^{-z_0 t}dt = Y(s)|_{s=z_0} = Y(z_0) = \frac{(z_0 - z_0)\bar{N}(z_0)}{z_0 \Delta_{CL}(z_0)} = 0 \tag{9}$$

Since z_0 is a real number, the exponential term inside the integral in (9) is always positive. The output signal has an initial value $y(0) = 0$ and a final value $y(\infty) = 1$, so $y(t)$ is not identically equal to 0. However, the results stated in (9) indicate that the area under the curve of $y(t)$ (weighted by a time-varying positive number) over all time is equal to 0. Therefore, the output signal $y(t)$ must take on negative values. This means that when the open-loop system has a right-half plane (non-minimum-phase) zero, the step response spends part of its time going in the wrong direction. This is generally known as a non-minimum-phase response[1] or an inverse response[2]. This inverse response always exists when the closed-loop system has a right-half plane zero. Since zeros of the open-loop forward transfer function $G(s)$ appear as closed-loop zeros, then whenever $G(s)$ has a non-minimum-phase zero, the system's step response will exhibit undershoot, taking on negative values.

D. Concluding Remarks

The non-minimum-phase (inverse) response in the step response is due to right-half plane zeros in $G(s)$. If there are two or more such zeros, the form of the inverse response becomes more complex. If there are m_0 right-half plane zeros in $G(s)$, then there will be m_0 values of $t > 0$ such that $y(t) = 0$. If m_0 is an odd integer, the output signal $y(t)$ will go negative, starting at $t = 0$, before it becomes positive. If m_0 is an even integer, then the output signal will start in the positive direction, then become negative, and then become positive again, repeating this pattern as appropriate based on the value of m_0.

The assumption in (1) that the system is Type 1 was made for convenience in the argument in the previous section that the output signal $c(t)$ is not identically zero. This is easy to justify since for a unity feedback, stable closed-loop system with one or more open-loop poles at the origin, the steady-state error for a step input is always 0. The system can be Type 0, or it can be Type 2 or higher, and the inverse response is still obtained. Regardless of the system Type, and regardless of whether the final value is positive or negative, the step response will still exhibit the inverse behavior described here.

Figure 1 illustrates this non-minimum-phase behavior for four different open-loop transfer functions. The transfer function $G_1(s)$ is Type 1 with one zero in the RHP. The step response goes negative first, then goes positive, ending with zero steady-state error. The transfer function $G_2(s)$ is also Type 1, but with two zeros in the RHP. The step response goes positive first, then goes negative and back to positive, ending with zero steady-state error. The transfer functions $G_3(s)$ and $G_4(s)$ are identical except for the sign of the gain. They are both Type 0 and have one zero in the RHP. Thus, the step response goes in the wrong direction first, then in the correct direction, ending with its appropriate final value. The open-loop and corresponding closed-loop transfer functions are

$$G_1(s) = \frac{-0.5\,(s-0.1)}{s\,(s+1)^2}, \qquad T_{CL-1} = \frac{-0.5\,(s-0.1)}{s^3 + 2s^2 + 0.5s + 0.05} \tag{10}$$

$$G_2(s) = \frac{0.5\,(s-0.1)^2}{s\,(s+1)^2}, \qquad T_{CL-2} = \frac{0.5\,(s-0.1)^2}{s^3 + 2.9s^2 + 0.9s + 0.005} \tag{11}$$

$$G_3(s) = \frac{-0.5\,(s-0.1)}{(s+1)^3}, \qquad T_{CL-3} = \frac{-0.5\,(s-0.1)}{s^3 + 3s^2 + 2.5s + 1.05} \tag{12}$$

$$G_4(s) = \frac{0.5\,(s-0.1)}{(s+1)^3}, \qquad T_{CL-4} = \frac{0.5\,(s-0.1)}{s^3 + 3s^2 + 3.5s + 0.95} \tag{13}$$

[1] *Control System Design*, G.C. Goodwin, S.F. Graebe, and M.E. Salgado, Prentice Hall, Upper Saddle River, NJ, 2001.

[2] *Multivariable Feedback Control: Analysis and Design*, S. Skogestad and I. Postlethwaite, John Wiley & Sons, Chichester, UK, 1996.

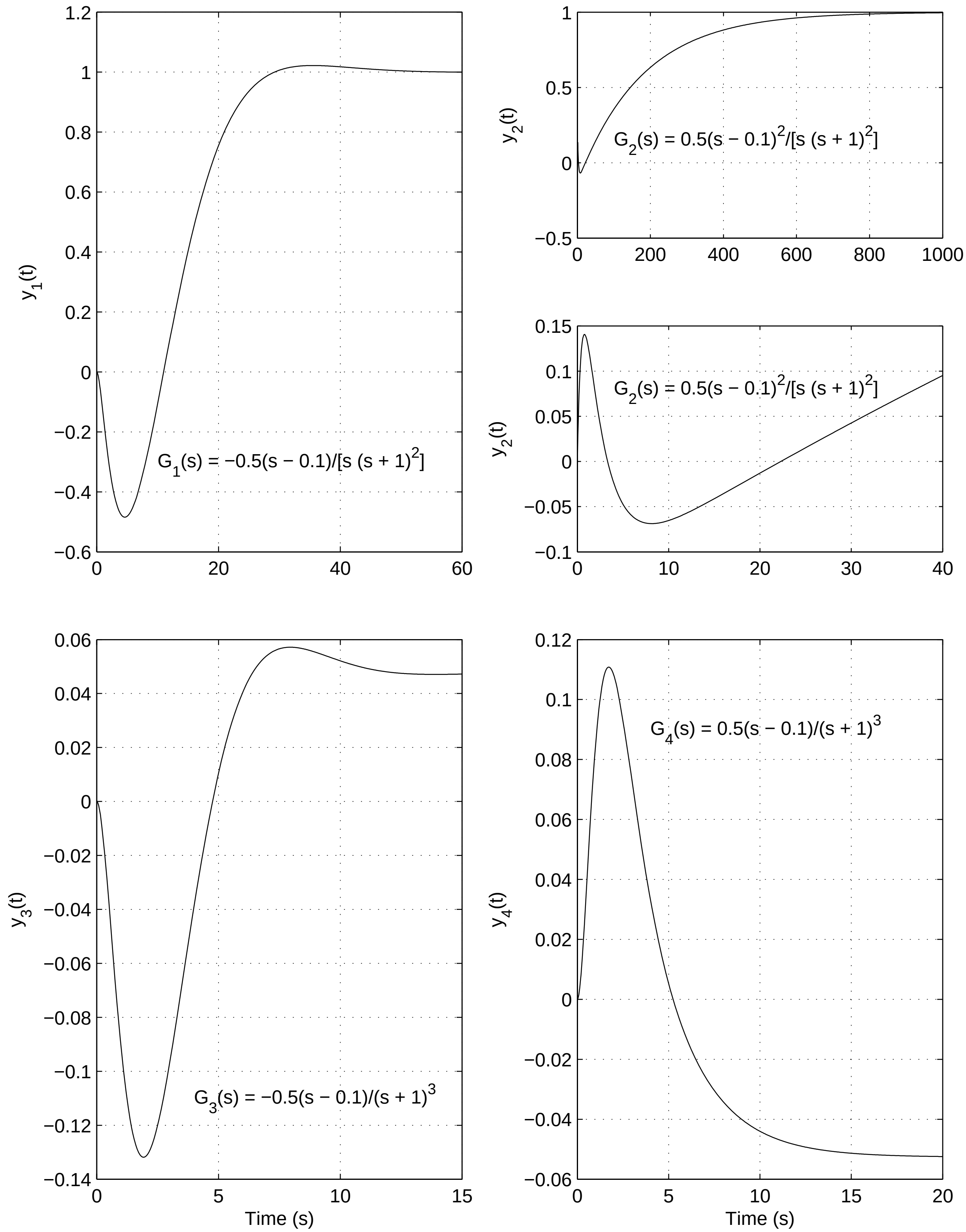

Fig. 1. Illustration of the effects of non-minimum-phase zeros on the closed-loop step response.