Robust Performance Example #1

The transfer function for a nominal system (plant) is given, along with the transfer function for one extreme system. These two transfer functions define a family of plants modeled by a multiplicative perturbation to the nominal system. The extreme plant is as far away from the nominal plant in terms of the frequency domain magnitude of the perturbation at each frequency as any system that must be considered. The goal is to design a compensator that provides both robust stability and robust performance for that family of systems.

Robust stability of the family means that the compensator provides internal closed-loop stability for each member of the family, and robust performance in this example means that each closed-loop system in the family satisfies a given frequency domain constraint on the sensitivity function $S\,(j\omega)$ [1]. The nominal and extreme plants are given in (1) by $G(s)$ and $G'(s)$, respectively.

$$G(s) = \frac{9}{s(s+2)}, \qquad G'(s) = \frac{5}{s(s+1)} \tag{1}$$

A lag-lead compensator is designed for the nominal system to place the dominant closed-loop poles at $s = -4 \pm j2$ and to make the steady-state error for a ramp input equal to $e_{ss} = 0.05$. Details on the root locus design procedures for transient response [2] and steady-state error [3] are available on the ECE 421 web site[1] and in most introductory texts on automatic control, for example [4]–[6]. A compensator that achieves these objectives is

$$K(s) = \frac{2.7745\,(s+0.04)\,(s+2.2654)}{(s+6.4075\cdot 10^{-3})\,(s+8.8284)} \tag{2}$$

This compensator is placed in series with the system to be controlled. If the different plant models represent physically different systems, instead of perturbations to a single nominal system, each system would be controlled by its own compensator, using the same compensator model given in (2). The nominal and extreme loop gains are

$$L(s) = K(s)G(s) = \frac{24.971\,(s+0.04)\,(s+2.2654)}{s\,(s+6.4075\cdot 10^{-3})\,(s+2)\,(s+8.8284)} \tag{3}$$

$$L'(s) = K(s)G'(s) = \frac{13.873\,(s+0.04)\,(s+2.2654)}{s\,(s+6.4075\cdot 10^{-3})\,(s+1)\,(s+8.8284)} \tag{4}$$

Performance will be defined by specifications imposed on the frequency domain magnitude of the sensitivity function, that is, on $|S(j\omega)| = |1 + L\,(j\omega)|^{-1}$. Nominal performance (NP) is defined by

$$\text{NP}: \qquad |S(j\omega)| < \frac{1}{|w_P\,(j\omega)|}, \qquad \forall\omega \qquad \Rightarrow \qquad |w_P\,(j\omega)\,S(j\omega)| < 1, \qquad \forall\omega \tag{5}$$

where $w_P\,(s)$ is a specified weighting function. The assumption here is that time domain specifications, such as overshoot and settling time, or frequency domain specifications, such as phase margin and bandwidth, can be achieved by proper choice of $w_P\,(s)$. The peak magnitude of $|S(j\omega)|$, the closed-loop bandwidth defined in terms of $|S(j\omega)|$, and the low-frequency slope of $|S(j\omega)|$ can be chosen through $w_P\,(s)$. The performance weighting function in this example is

$$w_P\,(s) = \frac{s/M + \omega_B}{s + \omega_B A} = \frac{0.5s + 0.8}{s} \tag{6}$$

which implies that $|S(j\omega)|$ should have at least a 20 db/decade slope at low frequencies (since $A = 0$), a maximum value of $M = 2$, and pass through -3 db at $\omega = \omega_B = 0.8$ rad/sec. For robust performance (RP), each member of the plant family must satisfy an expression like (5), that is,

$$\text{RP}: \qquad |S_i(j\omega)| < \frac{1}{|w_P\,(j\omega)|}, \qquad \forall\omega, \quad \forall S_i \in S_p \tag{7}$$

where S_p represents the sensitivities for the family of plants under consideration. Figure 1 shows the sensitivity magnitudes for the nominal and extreme systems and the magnitude of $1/\,|w_P\,(j\omega)|$. Since the sensitivity magnitudes are both below $1/\,|w_P\,(j\omega)|$ for all frequencies, each of those two systems satisfy the performance requirements. However, this does not imply RP, unless those are the only two plants in the family.

These notes are lecture notes prepared by Prof. Guy Beale for presentation in ECE 720, *Multivariable and Robust Control*, in the Electrical and Computer Engineering Department, George Mason University, Fairfax, VA. Additional notes can be found at: http://teal.gmu.edu/~gbeale/examples.html.

[1]See http://teal.gmu.edu/~gbeale/ece_421/examples_421.html

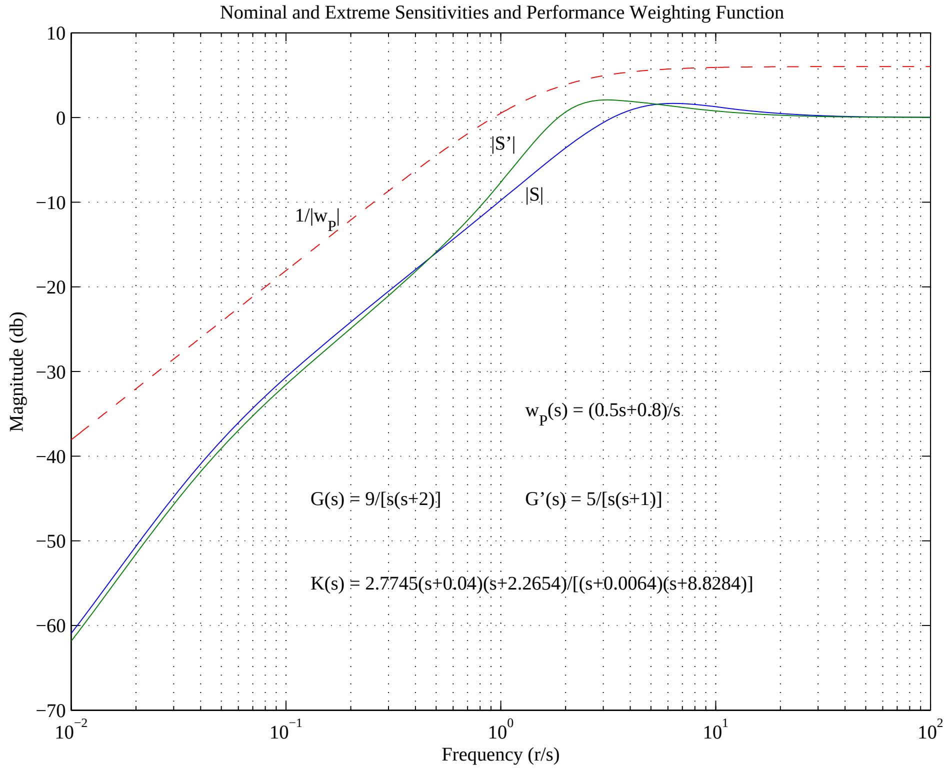

Fig. 1. Sensitivity magnitudes for the nominal and extreme plants, and the performance weighting function.

Before RP can be determined, stability must be investigated. Nominal stability (NS) is defined by the nominal closed-loop system being stable, that is, all the closed-loop poles of the compensator-nominal plant combination must be in the open left-half of the complex s-plane. Under the assumption of an unstructured multiplicative uncertainty model, robust stability (RS) is defined by the following constraint on the magnitude of the nominal complementary sensitivity function $|T(j\omega)|$.

$$\text{RS}: \qquad |T(j\omega)| < \frac{1}{|w_I(j\omega)|}, \qquad \forall\omega \qquad \Rightarrow \qquad |w_I(j\omega)\,T(j\omega)| < 1, \qquad \forall\omega \tag{8}$$

The nominal complementary sensitivity function is given by $T(s) = L(s)\,[1 + L(s)]^{-1}$, and $w_I(s)$ is the uncertainty weighting function that overbounds the magnitude of the maximum relative uncertainty $l_I(\omega)$, given by

$$l_I(\omega) = \max_{L_i \in L_p}\left[\frac{L_i(j\omega) - L(j\omega)}{L(j\omega)}\right] = \left[\frac{L'(j\omega) - L(j\omega)}{L(j\omega)}\right] = \left[\frac{G'(j\omega) - G(j\omega)}{G(j\omega)}\right] \tag{9}$$

$$l_I(s) = \left[\frac{G'(s) - G(s)}{G(s)}\right] = \left[\frac{\frac{5}{s(s+1)} - \frac{9}{s(s+2)}}{\frac{9}{s(s+2)}}\right] \tag{10}$$

$$l_I(s) = \frac{-0.4444s^2(s+2)(s-0.25)}{s^2(s+1)(s+2)} = \frac{-0.4444(s-0.25)}{(s+1)} \tag{11}$$

L_p represents the loop gains for all the members of the family of plants being considered. Since $G'(s)$ is an extreme plant and the only perturbed model given, the relative uncertainty can be computed based on that transfer function. Since only the frequency response magnitude $|w_I(j\omega)|$ is needed for overbounding $l_I(\omega)$, $w_I(s)$ can always be chosen to be a stable and minimum-phase transfer function. For this example, $w_I(s)$ is given by

$$w_I(s) = \frac{0.4444(s+0.25)}{(s+1)} = \frac{(0.4444s + 0.1111)}{(s+1)} \tag{12}$$

which provides an exact overbounding for $|l_I(\omega)|$. Since $|T(j\omega)| < 1/|w_I(j\omega)|$ is required for robust stability, the low frequency magnitude restriction on $|T(j\omega)|$ is $1/0.1111 = 9$ (19.1 db), and the high frequency restriction on $|T(j\omega)|$ is $1/0.4444 = 2.25$ (7.04 db). Figure 2 shows the nominal complementary sensitivity $|T(j\omega)|$ along with $1/|w_I(j\omega)|$. Since $|T(j\omega)| < 1/|w_I(j\omega)|$ for all frequencies, the family is robustly stable [1]. This means that any plant-compensator combination

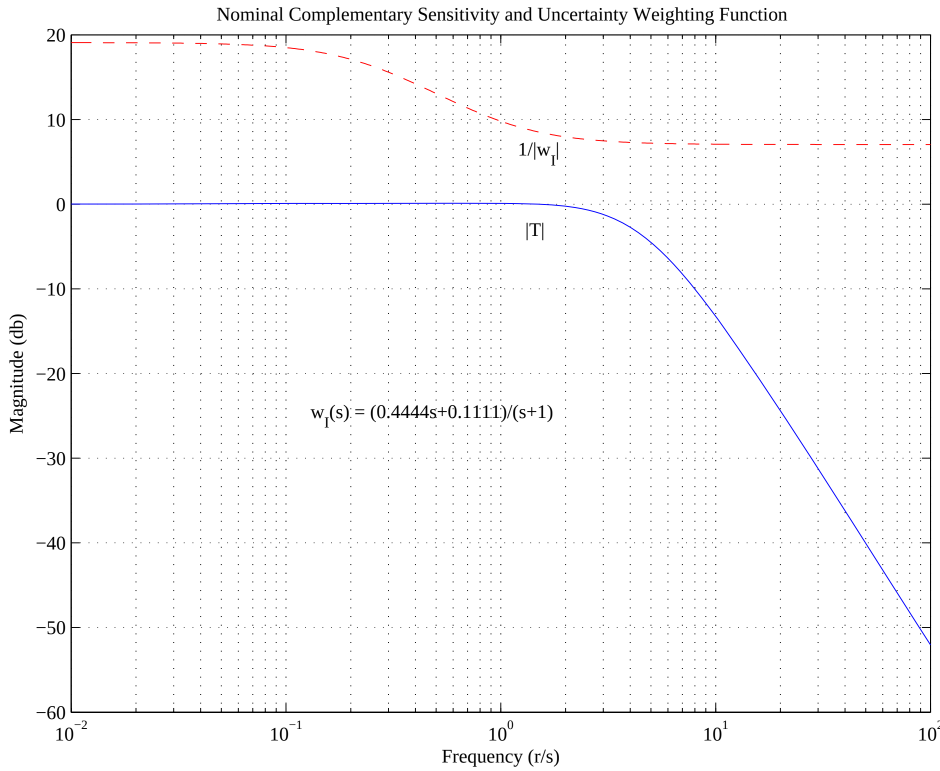

Fig. 2. Nominal complementary sensitivity and uncertainty weighting function.

whose polar plot at each frequency falls within the closed disk of radius $|w_I(j\omega)L(j\omega)|$ centered at $L(j\omega)$ will be closed-loop stable[2].

Now that nominal performance and robust stability have been achieved, the issue of robust performance can be addressed. The derivation of the condition for robust performance for the family of systems is illustrated in Fig. 3. For robust performance, the condition of (7) must be satisfied. From the figure it can be seen that the specified performance is represented at each frequency by a circle of radius $|w_P(j\omega)|$ centered at $-1+j0$, and the circle of uncertainty at each frequency is centered at $L(j\omega)$ with radius $|w_I(j\omega)L(j\omega)|$. The distance from the -1 point to the nominal loop gain is $|1+L(j\omega)|$. For robust performance the circle of uncertainty must not encircle or intersect the circle for performance at any frequency. Therefore, the distance $|1+L(j\omega)|$ must be greater than the sum of the radii of the circles representing uncertainty and performance, that is,

$$|w_P(j\omega)| + |w_I(j\omega)L(j\omega)| < |1+L(j\omega)|, \quad \forall\omega \tag{13}$$

$$\frac{|w_P(j\omega)|}{|1+L(j\omega)|} + \frac{|w_I(j\omega)L(j\omega)|}{|1+L(j\omega)|} < 1, \quad \forall\omega \tag{14}$$

$$|w_P(j\omega)S(j\omega)| + |w_I(j\omega)T(j\omega)| < 1, \quad \forall\omega \tag{15}$$

Since $|w_P(j\omega)S(j\omega)| < 1$ at all frequencies corresponds to NP and $|w_I(j\omega)T(j\omega)| < 1$ at all frequencies corresponds to RS, expression (15) shows that necessary conditions for RP are NP and RS. Therefore, the nominal system satisfying the performance requirements and every plant in the family being closed-loop stable are necessary conditions for every plant in the family satisfying the performance requirements. However, (15) also shows that by themselves NP and RS are not sufficient conditions for RP. The sum of those two terms must also be less than 1 at all frequencies in order to achieve robust performance. Figure 4 shows the sum of the weighted sensitivity and weighted complementary sensitivity magnitudes for this example. Since the peak value of that sum is shown to be less than 1, this family of plants with the compensator given in (2) has robust performance. Therefore, the design goal for the compensator has been achieved.

Figure 5 shows the closed-loop step responses and the open-loop polar plots for the nominal and extreme systems whose loop gains are given in (3) and (4), respectively. The nominal system clearly has better performance in terms of overshoot, and it also has a shorter settling time that might be considered a benefit. However, the performance of the perturbed (extreme) system also satisfies the specification on the sensitivity function, so that performance must also be considered acceptable.

[2]See the example Robust Stability #1 at: http://teal.gmu.edu/~gbeale/ece_720/examples_720.html.

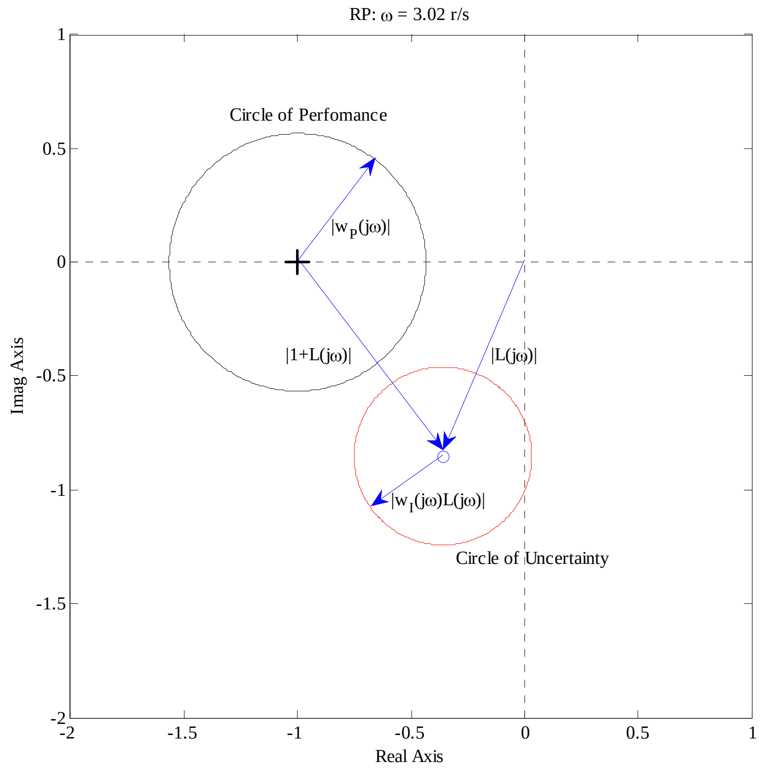

Fig. 3. Illustration of the robust performance requirement: $|w_P(j\omega)| + |w_I(j\omega) L(j\omega)| < |1 + L(j\omega)|, \quad \forall \omega$

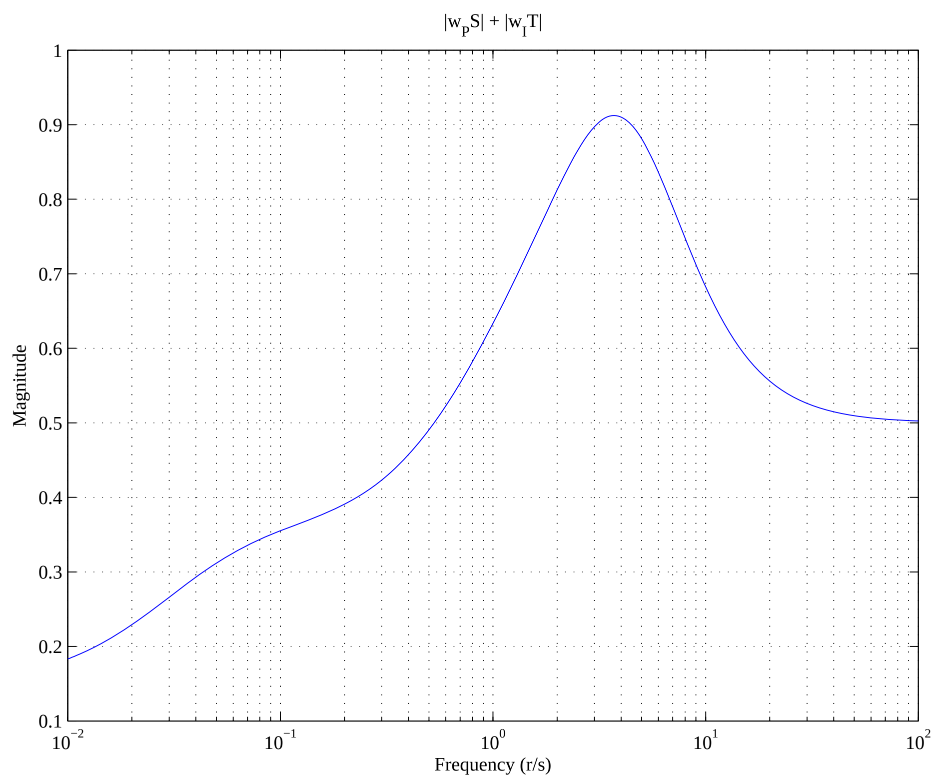

Fig. 4. Sum of weighted sensitivity and weighted complementary sensitivity functions.

Although the nominal system has a larger phase margin than the perturbed system, both of those systems have phase margins that generally would be considered acceptable.

Figure 6 shows the maximum and minimum magnitudes for the sensitivity and complementary sensitivity functions for any plant model allowed in the family of systems defined by $w_I(s)$. The maximum and minimum magnitudes for the sensitivity function are given by

$$|S(j\omega)|_{\max} = \frac{1}{|1+L(j\omega)| - |w_I(j\omega)L(j\omega)|}, \qquad |S(j\omega)|_{\min} = \frac{1}{|1+L(j\omega)| + |w_I(j\omega)L(j\omega)|} \tag{16}$$

and the maximum and minimum magnitudes for the complementary sensitivity function are

$$|T(j\omega)|_{\max} = \left|\frac{L(j\omega)}{1+L(j\omega)}\right|_{\max}, \qquad |T(j\omega)|_{\min} = \left|\frac{L(j\omega)}{1+L(j\omega)}\right|_{\min} \tag{17}$$

The minimum and maximum sensitivity magnitudes are easy to compute based on the geometric relationship between sensitivity and the circles of uncertainty. The corresponding complementary sensitivity magnitudes are less easy to visualize since they depend on the ratios of vectors drawn from the origin and from the -1 point to $L(j\omega)$. Knowing that the family of systems is robustly stable (from $\|w_I T\|_\infty < 1$), the fact that the maximum sensitivity magnitude for all systems in the family is below $1/|w_P(j\omega)|$ at all frequencies indicates that the family also has robust performance. In other words, if all plants in the family are stable, and the worst-case system satisfies the performance criterion, then all systems in the family satisfy the performance criterion and robust performance is achieved.

The nine graphs in Fig. 7 illustrate the relationship between the circle of uncertainty at a particular frequency, centered on $L(j\omega)$ with radius $|w_I(j\omega)L(j\omega)|$, and the performance weighting function at the same frequency, centered at $-1+j0$ with radius $|w_P(j\omega)|$. For the family of systems to be robustly stable, the circles of uncertainty must not intersect the circles representing performance at any frequency. This relationship cannot conveniently be shown on a single plot since the circles must be compared on a frequency-by-frequency basis. At low frequencies, $|w_P(j\omega)| \gg 1$, and those circles would clearly enclose the circles of uncertainty at high frequencies. To show both sets of circles at all frequencies on a single plot and to indicate which pairs of circles should be compared would be difficult. The graphs in Fig. 7 merely illustrate how the circles of uncertainty move along the polar plot, decreasing in radius, and slide past the circles of the performance weighting function when the family has robust performance. The plot of $|w_P(j\omega)S(j\omega)| + |w_I(j\omega)T(j\omega)|$ in Fig. 4 is the easiest way to graphically show whether or not a family of systems is robust in terms of performance.

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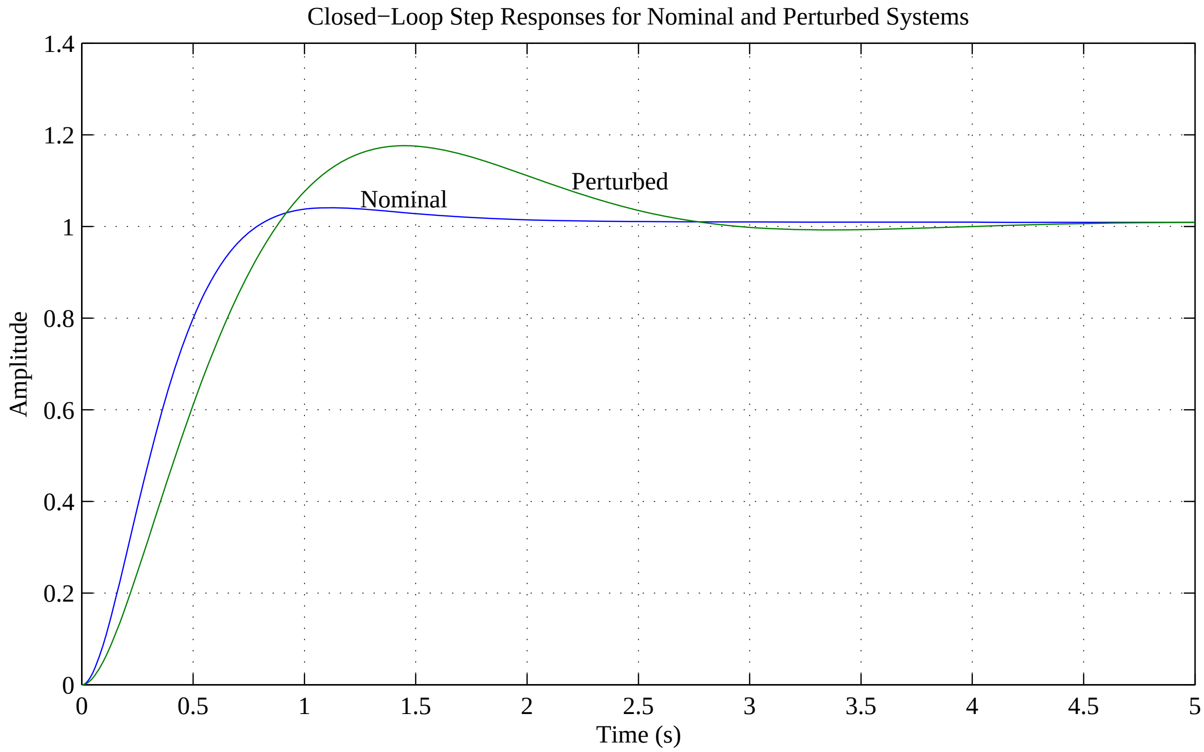

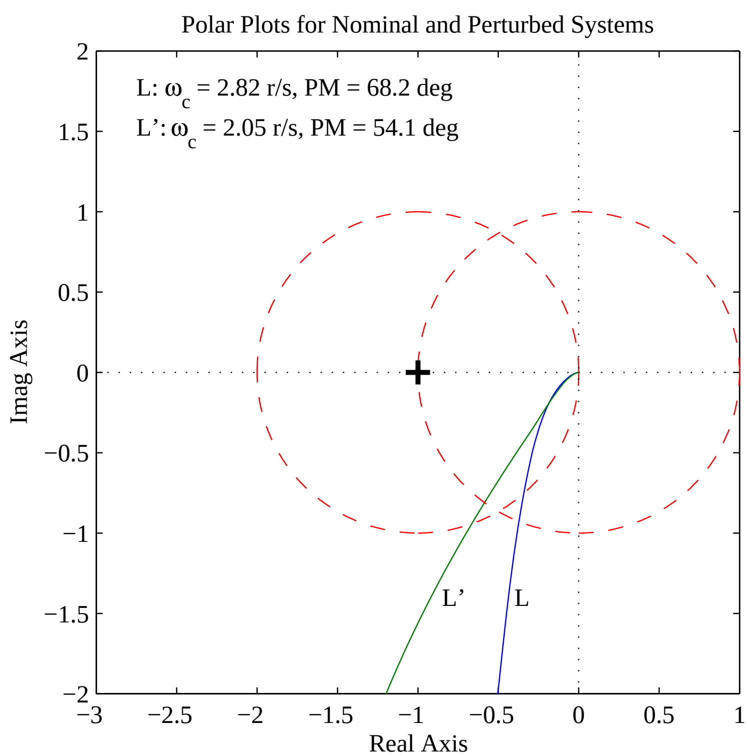

Fig. 5. Closed-loop step responses and open-loop polar plots for the nominal and extreme systems.

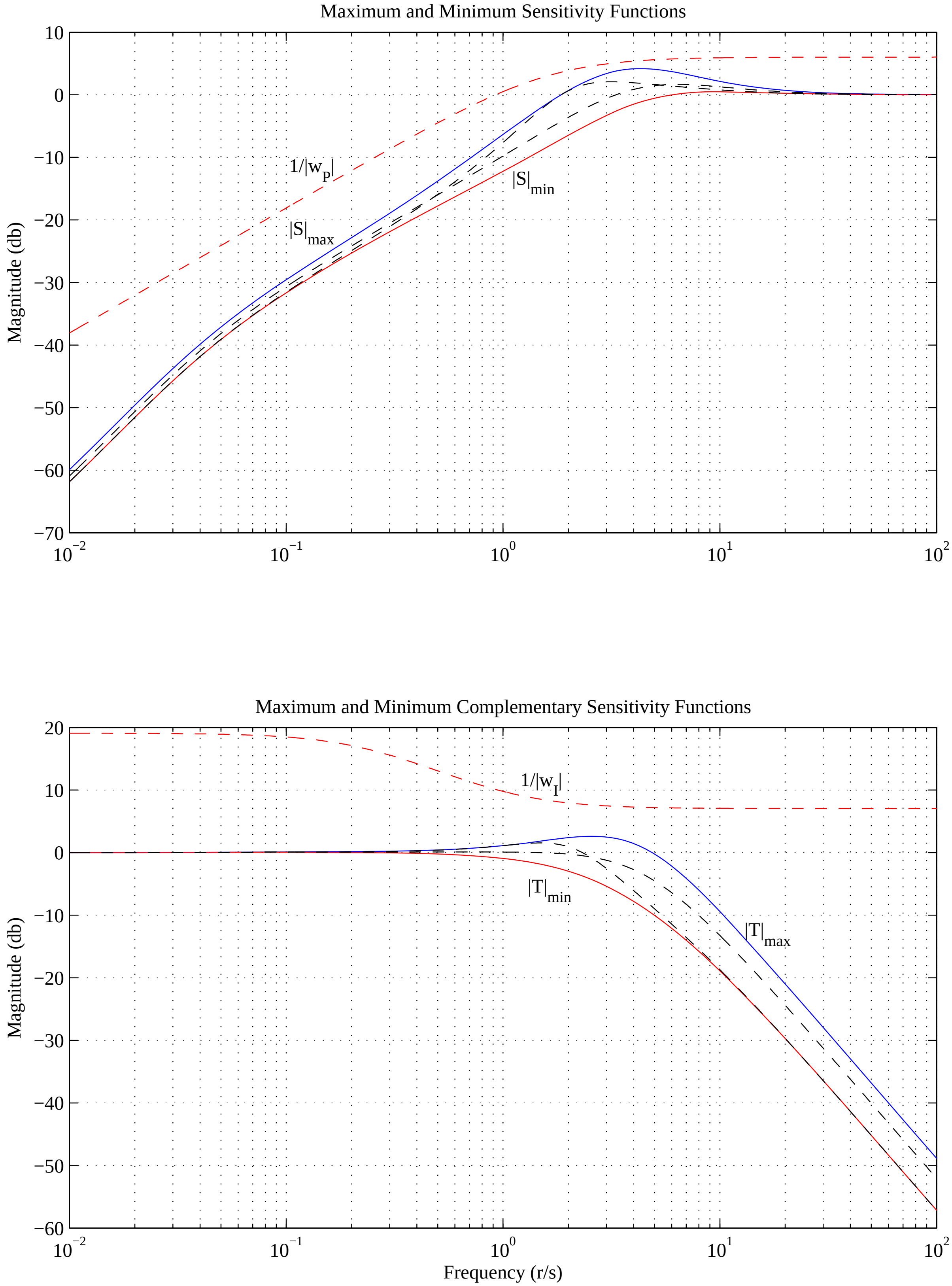

Fig. 6. Worst case sensitivity and complementary sensitivity functions.

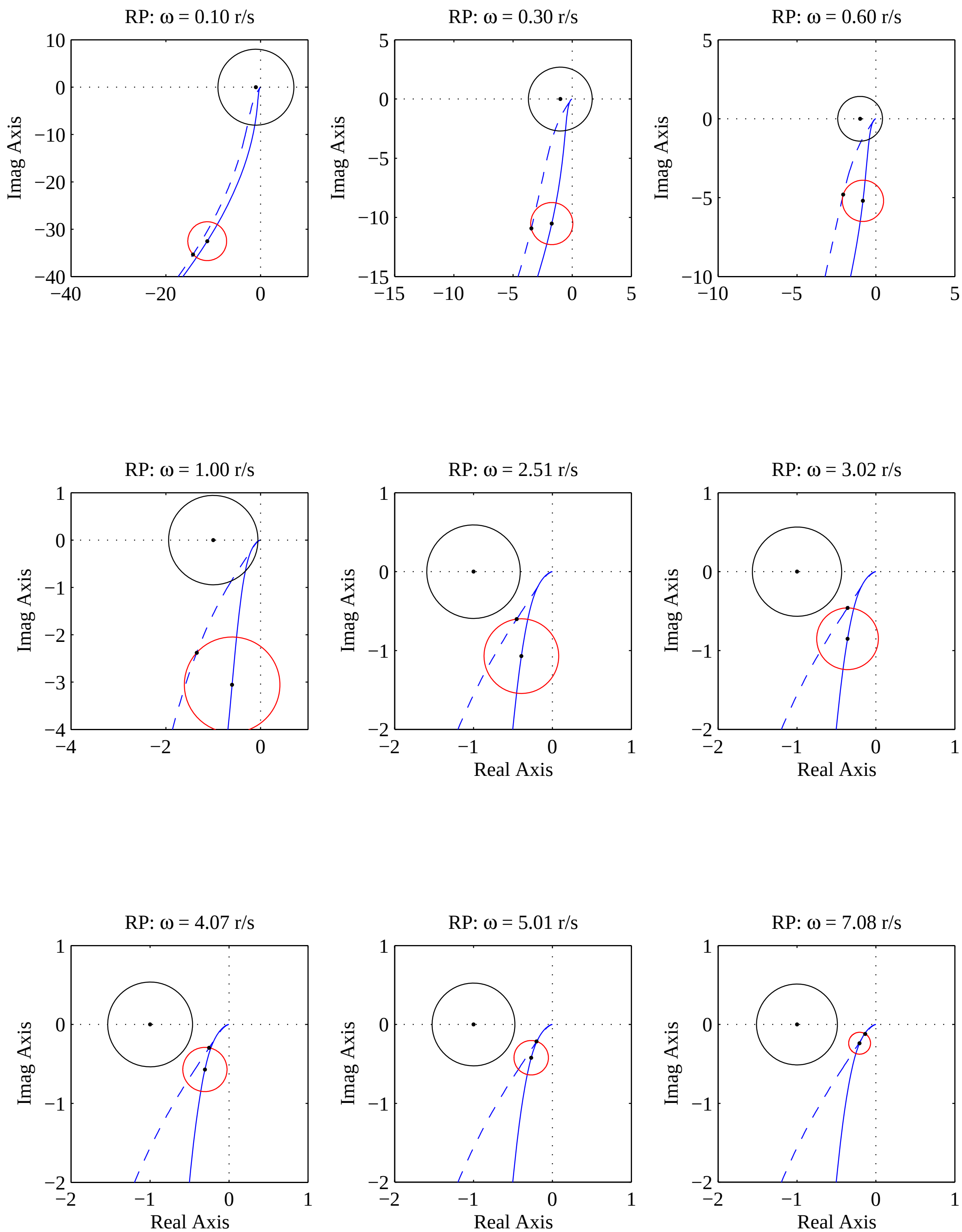

Fig. 7. Illustration of the relationship between the circles of uncertainty and the circles of performance for a family of systems with robust performance.